A NOTE ON QUASI-OPEN MAPS

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ABSTRACT. Let $f : X \to Y$ be quasi-open. We show that: (1) If $A \subseteq X$ is open, $f|A$ is quasi-open, (2) $f : X \to f(X)$ is quasi-open. (3) And let $f_\alpha : X_\alpha \to Y_\alpha$ be quasi-open. Then $\Pi f_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$, defined by $\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$, is quasi-open. (4) Lastly, if $f_i : X_i \to Y$ are quasi-open, $i = 1, 2$, then $F : X_1 \oplus X_2 \to Y$, defined by $F(x) = f_i(x)$, $x \in X_i$, is also quasi-open.

1. Introduction

The concept of a quasi-open map was introduced by Kao[3] in 1983. Some characterizations of $M_1$-spaces, in terms of quasi-open maps, have been given by Kao[3].

The continuous maps and the quasi-open maps are not related. See the Examples of this note. But the quasi-open maps have the properties which are similar to those of the continuous maps. The purpose of this note is to derive the characterizations of quasi-open maps.

Let $X$, $Y$ and $Z$ be topological spaces with no separation axioms assumed unless explicitly stated.

The interior of a subset $U$ of $X$ will be denoted by $\text{Int}(U)$. Notations and terminologies not explained here but used in this note are taken from Dugundji[2].

2. Results

Definition 1 [3]. A mapping $f : X \to Y$ is called quasi-open if $\text{Int}(f(U)) \neq \emptyset$ for every non-empty open subset $U \subseteq X$. 

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Example 1. Let \( X = \{a, b, c\} \) and \( \tau = \emptyset, \{a\}, \{b, c\}, X \) be a topology on \( X \). Let \( Y = \{p, q\} \), and \( \sigma = \emptyset, \{p\}, Y \) be a topology on \( Y \). Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = p \) and \( f(b) = f(c) = q \). Then \( f \) is continuous but not quasi-open.

Example 2. Let \( X = \{a, b, c\} \) and \( \tau = \emptyset, \{a\}, \{a, b\}, X \) be a topology on \( X \). Let \( Y = \{p, q, r\} \) and \( \sigma = \emptyset, \{p\}, \{p, r\}, \{q, r\}, Y \) be a topology on \( Y \). Define \( g : (X, \tau) \rightarrow (Y, \sigma) \) by \( g(a) = p \), \( g(b) = q \), \( g(c) = r \). Then \( g \) is quasi-open but not continuous.

Lemma 1 [3]. If \( f : X \rightarrow Y \) is open, \( f \) is quasi-open.

Lemma 2(cf. [1]). If \( f : X \rightarrow Y \) is local homeomorphism, \( f \) is quasi-open.

Proof. Every local homeomorphism is a countinuous open map[1]. By Lemma 1, \( f \) is quasi-open.

Let \( \Pi X_\alpha \) be the product space with the product topology. Then the \( \beta \)-th projection map \( \pi_\beta : \Pi X_\alpha \rightarrow X_\beta \) is continuous, open and surjective. Hence, by Lemma 1, we obtain the following lemma.

Lemma 3 [3]. \( \pi_\beta \) is quasi-open.

Lemma 4 [Composition](cf. [3]). Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be quasi-open maps. Then \( g \circ f \) is quasi-open.

Proof. Let \( U \) be any non-empty open set in \( X \). Since \( f \) is quasi-open, \( \text{Int}(f(U)) \neq \emptyset \). Since \( g \) is also quasi-open, \( \text{Int}(g(\text{Int}(f(U)))) \neq \emptyset \).

But we know that \( \text{Int}(g(\text{Int}(f(U)))) \subset \text{Int}(g(f(U))) \). Hence \( \text{Int}(g(f(U))) \neq \emptyset \). This completes the proof.

If \( A \) is an open subset of \( X \), then the inclusion \( i : A \rightarrow X \) is open [2].

Proposition 5 [Restriction of Domain]. Let \( f : X \rightarrow Y \) be a quasi-open map, and \( A \) open subspace of \( X \). Then \( f|A : A \rightarrow Y \) is quasi-open.

Proof. Let \( U \) be any non-empty open set in \( A \). Then \( U \) is a non-empty open set in \( X \). We know that \( f|A = f \circ i \), where \( i : A \rightarrow X \) is an inclusion, and that \( i \) is open [2]. By Lemma 4, we get the result.

Proposition 6 [Restriction of Range]. If \( f : X \rightarrow Y \) is quasi-open and \( f(X) \) is taken the subspace topology, then \( f : X \rightarrow f(X) \) is quasi-open.
Proof. Let $U$ be any non-empty open set in $X$. Then $\text{Int}_{f(X)}(f(U)) \supset \text{Int}_Y(f(U)) \cap f(X) = \text{Int}_Y(f(U))$. Since $f$ is quasi-open, $\text{Int}_Y(f(U)) \neq \emptyset$. Hence $f : X \to f(X)$ is quasi-open.

Proposition 7. Let $f_\alpha : X_\alpha \to Y_\alpha$ be onto for each $\alpha$. Define $\Pi f_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$ by $\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$. If $f_\alpha$ is quasi-open for each $\alpha$, $\Pi f_\alpha$ is quasi-open.

Proof. Let $U = U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \Pi_{\alpha \neq \alpha_1} \alpha X_\alpha$ be a non-empty basic open set in $\Pi X_\alpha$. Then $\text{Int}(\Pi f_\alpha(U)) = \text{Int}(f_{\alpha_1}(U_{\alpha_1})) \times \cdots \times \text{Int}(f_{\alpha_n}(U_{\alpha_n})) \times \Pi_{\alpha \neq \alpha_1} \alpha Y_\alpha$ is non-empty. Hence $\Pi f_\alpha$ is quasi-open.

Let $X_1 \oplus X_2$ be a sum of disjoint topological spaces $X_1$ and $X_2$. Define $F : X_1 \oplus X_2 \to Y$ by $F(x) = f_i(x)$ if $x \in X_i$, where $f_i : X_i \to Y$, $i = 1, 2$.

Proposition 8. If $f_i : X_i \to Y$ are quasi-open, $i = 1, 2$, $F$ is quasi-open.

Proof. Let $U$ be any non-empty open set in $X_1 \oplus X_2$. Then $U \cap X_i$ are open in $X_i$ by the definition of a topological sum. Since $f_i$ are quasi-open, $\emptyset \neq \text{Int}(f_1(U \cap X_1)) \cup \text{Int}(f_2(U \cap X_2)) \subset \text{Int}(f_1(U \cap X_1) \cup f_2(U \cap X_2)) = \text{Int}(F(U))$. Hence $F$ is quasi-open.

Corollary 9. If $f_i : X_i \to Y$ are quasi-open for $i = 1, 2, \ldots, n$, the map $F : \oplus_{i=1}^n X_i \to Y$, defined by $F(x) = f_i(x)$ if $x \in X_i$, is quasi-open.

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