A NOTE ON MINIMAL SETS OF THE CIRCLE MAPS

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ABSTRACT. For continuous maps $f$ of the circle to itself, we show that (1) every $\omega$-limit point is recurrent (or almost periodic) if and only if every $\omega$-limit set is minimal, (2) every $\omega$-limit set is almost periodic, then every $\omega$-limit set contains only one minimal set.

1. Introduction

Let $I$ be the unit interval, $S^1$ the circle and $X$ a topological space. And let $C^0(X, X)$ denote the set of continuous maps from $X$ into itself.

Let $f \in C^0(X, X)$. For any positive integer $n$, we define $f^n$ inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let $f^0$ denote the identity map of $X$.

For any $f \in C^0(X, X)$, let $P(f), AP(f), R(f), \Lambda(f)$ and $\Omega(f)$ denote the collection of the periodic points, almost periodic points, recurrent points, $\omega$-limit points and nonwandering points of $f$, respectively.

$Y \subset X$ is called an invariant subset of $f$ if $f(Y) \subset Y$; and strongly invariant if $f(Y) = Y$. Suppose $Y \subset X$ is non-void, closed, and invariant relative to $f$. If $Y$ has no proper subset which is non-void and invariant relative to $f$ then $Y$ is said to be a minimal set of $f$.

In 1986, J.C.Xiong [5] proved that for any interval map $f$, every $\omega$-limit point is recurrent (or almost periodic) if and only if every $\omega$-limit set is minimal. We have the same result for map of the circle.

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Theorem 5. Let $f \in C^0(S^1, S^1)$. Then the followings are equivalent.

1. $\Lambda(f) = AP(f)$.
2. $\Lambda(f) = R(f)$.
3. For every $x \in S^1$, the $\omega$-limit set $\omega(x, f)$ of $x$ is minimal.

In 1966, A.N.Sarkovskii [3] showed that for any interval map $f$, the following conditions (i) and (ii) are equivalent.

(i) The periods of all periodic points of $f$ are powers of 2.

(ii) For every $x \in I$ either the $\omega$-limit set $\omega(x)$ of $x$ is a periodic orbit of $f$ or the set $\omega(x)$ contains no periodic orbit of $f$.

In [5], J.C.Xiong showed that the condition (i) is equivalent to the following condition (iii).

(iii) For every point $x \in I$, the $\omega$-limit set $\omega(x, f)$ of $x$ contains only one minimal set.

In this paper, we will prove the following theorem.

Theorem 8. Let $f \in C^0(S^1, S^1)$. Suppose that $\Lambda(f) = AP(f)$. Then we have that for any $x \in S^1$, the $\omega$-limit set $\omega(x, f)$ of $x$ contains only one minimal set.

2. Preliminaries and definitions

Let $(X, d)$ be a metric space and $f \in C^0(X, X)$. The forward orbit $O(x)$ of $x \in X$ is the set $\{f^k(x) \mid k = 1, 2, \ldots \}$. A point $x \in X$ is called a periodic point of $f$ if for some positive integer $n$, $f^n(x) = x$. The period of $x$ is the least such integer $n$. We denote the set of periodic points of $f$ by $P(f)$.

A point $x \in X$ is called a recurrent point of $f$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of $f$ by $R(f)$.

A point $x \in X$ is called a nonwandering point of $f$ if for every neighborhood $U$ of $x$, there exists a positive integer $m$ such that $f^m(U) \cap U \neq \emptyset$. We denote the set of nonwandering points of $f$ by $\Omega(f)$.

A point $x \in X$ is called a almost periodic point of $f$ if for any $\epsilon > 0$ one can find an integer $N > 0$ with the following property that for any integer $q > 0$ there exists an integer $r$, $q \leq r < q + N$, such that $d(f^r(x), x) < \epsilon$, where $d$ is the metric of $X$. We denote the set of almost periodic points of $f$ by $AP(f)$. 
A point \( y \in X \) is called an \( \omega \)-limit point of \( x \) if there exists a sequence \( \{n_i\} \) of positive integers with \( n_i \to \infty \) such that \( f^{n_i}(x) \to y \). We denote the set of \( \omega \)-limit points of \( x \) by \( \omega(x, f) \). Define \( \Lambda(f) = \bigcup_{x \in X} \omega(x, f) \).

3. Main Results

**Lemma 1[4].** Let \( f \in C^0(S^1, S^1) \). Then we have that \( x \in AP(f) \) if and only if \( x \in \omega(x, f) \) and \( \omega(x, f) \) is minimal.

The following lemma follows from [1].

**Lemma 2.** Let \( f \in C^0(S^1, S^1) \). Then we have

\[
P(f) \subset AP(f) \subset R(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).
\]

The following lemma found in [2]

**Lemma 3.** Let \( f \in C^0(S^1, S^1) \) and let \( P(f) \) be empty. Then we have \( \Omega(f) = R(f) \).

**Corollary 4.** Let \( f \in C^0(S^1, S^1) \) with \( P(f) \neq \phi \). Then the followings are equivalent.

1. \( \overline{R(f)} = P(f) \).
2. \( \Omega(f) = P(f) \).
3. \( \Lambda(f) = P(f) \).

**Theorem 5.** Let \( f \in C^0(S^1, S^1) \). Then the following conditions are equivalent.

1. \( \Lambda(f) = AP(f) \).
2. \( \Lambda(f) = R(f) \).
3. For every \( x \in S^1 \), the \( \omega \)-limit set \( \omega(x, f) \) of \( x \) is minimal.

*Proof.* (1) \( \Rightarrow \) (2): Obvious by Lemma 2.

(2) \( \Rightarrow \) (3): Let \( x \) be arbitrary point in \( S^1 \), and let \( y \) be any point in \( \omega(x, f) \). Then there exists a sequence \( n_i \to \infty \) such that \( f^{n_i}(x) \to y \). Suppose that \( z \in \omega(y, f) \). Then there exists a sequence \( m_i \to \infty \) such that \( f^{m_i}(y) \to z \). Therefore \( f^{m_i+n_i}(x) \to z \), and hence \( z \in \omega(x, f) \). Thus \( \omega(y, f) \subset \omega(x, f) \). Now we show that \( \omega(x, f) \subset \omega(y, f) \). Since \( y \) is arbitrary point in \( \omega(x, f) \), it suffices to show that \( y \in \omega(y, f) \). We know that \( y \in \Lambda(f) \) by definition. By assumption, \( y \in R(f) \), and hence \( y \in \omega(y, f) \). Therefore \( \omega(x, f) \) is minimal for any \( x \in S^1 \).
(3) $\Rightarrow$ (1): Suppose that $\omega(x, f)$ is minimal for any $x \in S^1$. Let $y \in \Lambda(f) \setminus AP(f)$. Then there exists $z \in S^1$ such that $y \in \omega(z, f)$. Since $\omega(z, f)$ is minimal, $\omega(y, f) = \omega(z, f)$. Hence $y \in \omega(y, f)$ and $\omega(y, f)$ is minimal, and hence $y \in AP(f)$ by Lemma 1. This is a contradiction.

**Corollary 6.** Let $f \in C^0(S^1, S^1)$. Suppose that $P(f)$ is closed. Then the followings are equivalent.

1. $R(f) = AP(f)$.
2. $\Omega(f) = AP(f)$.
3. $\Lambda(f) = AP(f)$.
4. $\Lambda(f) = R(f)$.
5. For every $x \in S^1$, the $\omega$-limit set $\omega(x, f)$ of $x$ is minimal.

**Corollary 7.** Let $f \in C^0(S^1, S^1)$. Suppose that for any $x \in S^1$, the $\omega$-limit set $\omega(x, f)$ of $x$ contains a minimal set containing $x$. Then we have $\Lambda(f) = AP(f)$.

**Theorem 8.** Let $f \in C^0(S^1, S^1)$. Suppose that $\Lambda(f) = AP(f)$. Then we have that for any $x \in S^1$, the $\omega$-limit set $\omega(x, f)$ of $x$ contains only one minimal set.

**Proof.** Suppose that $\Lambda(f) = AP(f)$. Let $x \in S^1$. Assume that there exist two minimal sets $M, N$ with $M \subset \omega(x, f)$ and $N \subset \omega(x, f)$. Then for every $a \in M$ and $b \in N$, $M = \omega(a, f)$ and $N = \omega(b, f)$. We know that the $\omega$-limit set $\omega(x, f)$ of $x$ is minimal by Theorem 5. Since $a, b \in \omega(x, f)$,

$$M = \omega(a, f) = \omega(x, f) = \omega(b, f) = N.$$ 

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