OPERATORS ON GENERALIZED BLOCH SPACE

KI SEONG CHOI AND GYE TAK YANG

ABSTRACT. In [5], Zhu introduces a bounded operator $T$ from $L^\infty(D)$ into Bloch space $B$. In this paper, we will consider the generalized Bloch spaces $B_q$ and find bounded operator from $L^\infty(D)$ into $B_q$.

1. Introduction

Let $\mathbb{C}$ be the complex number plane and $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ be the open unit disk in $\mathbb{C}$. Let $dA(z)$ be the area measure on $D$ normalized so that the area is 1. For $1 \leq p < \infty$, $L^p(D, dA)$ will denote the Banach space of Lebesgue measurable functions $f$ on $D$ with

$$\left[ \int_{D} |f(z)|^p dA(z) \right]^{\frac{1}{p}} \leq \infty.$$  

$L^\infty(D, dA)$ will denote the Banach space of Lebesgue measurable functions $f$ on $D$ with

$$\text{esssup}\{ |f(z)| : z \in D \} < \infty.$$  

The Bloch space of $D$, denoted by $B$, consists of analytic functions $f$ on $D$ such that $\sup\{(1 - |z|^2)|f'(z)| : z \in D\} < \infty$. The set $B$ of Bloch functions (modulo constant functions) become a Banach space ([1], p.13). In [5], Zhu show that the integral operator $T$ which is represented by

$$Tf(z) = \int_{D} \frac{f(w)}{(1 - zw)^2} dA(w)$$

is a bounded operator from $L^\infty(D)$ into $B$.  

Received by the editors Oct. 23, 1997 and, in revised form Feb. 3, 1998.  
1991 Mathematics Subject Classifications. Primary 32H25, 32E25.  
Key words and phrases. Generalized Bloch space.
For each $q > 0$, the space $B_q$ consist of analytic functions $f$ on $D$ with the property that

$$\sup\{(1 - |z|^2)^q |f'(z)| : z \in D\} < \infty.$$ 

For each $q > 0$, let $T_q$ denote the operator defined by

$$T_q f(z) = q \int_D \frac{f(w)}{(1 - zw)^{1+q}} dA(w), \quad z \in D.$$ 

In this paper, we will show that generalized Bloch spaces $B_q$ are Banach spaces. Also we will investigate some properties of $T_q$. In particular, we will show that $T_q$ is a bounded operator from $L^\infty(D)$ into $B_q$.

2. $B_q$ is a Banach space

Let us define a norm on $B_q$ as follows;

$$\|f\|_q = |f(0)| + \sup\{(1 - |z|^2)^q |f'(z)| : z \in D\}.$$ 

**Lemma 1.** If $f \in B_q$, $q > 0$, then

$$|f(z)| \leq |f(0)| + \|f\|_q (1 - |z|^2)^{-q}.$$ 

**Proof.**

$$|f(z) - f(0)| \leq \int_0^1 |f'(tz)||z|dt$$

$$\leq \int_0^1 \frac{|f'(tz)|(1 - |tz|^2)^q}{(1 - |tz|^2)^q}dt$$

$$\leq \|f\|_q \int_0^1 \frac{1}{(1 - t|z|^2)^q}dt$$

$$\leq \|f\|_q \frac{1}{(1 - |z|^2)^q},$$

since the first inequality follows from the followings

$$f(z) - f(0) = \int_0^1 f'(tz)zdt.$$ 

Thus the desired result follows. \qed
Theorem 1. For each $q > 0$, $B_q$ is a Banach space with norm $\|\cdot\|_q$.

Proof. Let $(f_n)$ be a Cauchy sequence in $B_q$. By Lemma 1,

$$|(f_n - f_m)(z) - (f_n - f_m)(0)| \leq \|f_n - f_m\|_q (1 - |z|^2)^{-q}.$$ 

It follows that the sequence $(f_n)$ is a Cauchy sequence in the topology of uniform convergence on compact sets. Thus there exists holomorphic function $f : D \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of $D$ as $n \to \infty$.

Since $f_n \to f$ uniformly on compact subsets of $D$ as $n \to \infty$, it follows that $f'_n(z) \to f'(z)$ uniformly on compact subsets of $D$ as $n \to \infty$.

Thus, for each $n$

$$(1 - |z|^2)^q |(f_n - f_m)'(z)| \to (1 - |z|^2)^q |(f_n - f)'(z)| \text{ as } m \to \infty$$

for each $z \in D$. Therefore, for each sufficiently large $n$,

$$(1 - |z|^2)^q |(f_n - f)'(z)| \leq \varepsilon.$$

Namely, $\|f_n - f\|_q \leq \varepsilon$. 


3. Operator $T_q$ on $L^\infty(D)$

In the sequel, $C_0(D)$ is the space of complex-valued continuous functions on $D$ which vanish on the boundary.

Theorem 2. If $P$ is a polynomial, then there exists $f$ in $C_0(D)$ such that $P = T_q f$.

Proof. It suffices to show that $T_q g(z) = z^n$ for some $g \in C_0(D)$. In fact, if we consider the function $g(z) = (1 - |z|^2)^2 z^n$, then

$$T_q g(z) = q \int_D \frac{(1 - |w|^2)w^n}{(1 - z\overline{w})^{1+q}} dA(w)$$

$$= q \int_D (1 - |w|^2)w^n \sum_{k=0}^{\infty} \frac{\Gamma(k+1+q)}{k!\Gamma(1+q)} (z\overline{w})^k dA(w)$$

$$= q \sum_{k=0}^{\infty} z^k \frac{\Gamma(k+1+q)}{k!\Gamma(1+q)} \int_0^{2\pi} \int_0^1 (1 - r^2)^{n+k+1} e^{i(n-k)\theta} dr d\theta$$

$$= qz^n \frac{\Gamma(n+1+q)}{n!\Gamma(1+q)} \int_0^1 (1 - r^2)^{n+1} dr$$

$$= q \frac{\Gamma(n+1+q)}{n!\Gamma(1+q)} \frac{1}{(n+1)(n+2)(n+3)} z^n.$$
In third equality, if \( n \neq k \),
\[
\int_0^{2\pi} \int_0^1 (1-r^2)^{n+k+1} e^{i(n-k)\theta} \, dr \, d\theta = 0.
\]
Thus the desired result follows. \( \square \)

**Theorem 3.** For each \( q > 0 \), the operator \( T_q \) maps each function of the form \( z^n \bar{z}^m \) to a monomial where \( n \) and \( m \) are positive integers such that \( n \geq m \).

**Proof.**

\[
T_q(z^n \bar{z}^m) = q \int_D (1 - z \bar{z})^{1+q} dA(w)
= q \int_D w^n \bar{w}^m \sum_{k=0}^{\infty} \frac{\Gamma(k+1+q)}{k! \Gamma(1+q)} z^k \bar{z}^k \, dA(w)
= q \sum_{k=0}^{\infty} \frac{\Gamma(k+1+q)}{k! \Gamma(1+q)} \int_D w^n \bar{w}^{m+k} \, dA(w)
= q \frac{\Gamma(n-m+1+q)}{(n-m)! \Gamma(1+q)} \int_D w^n \bar{w}^n \, dA(w)
= q \frac{\Gamma(n-m+1+q)}{(n-m)! \Gamma(1+q)} \frac{1}{(n+1)(n+2)(n+3)} z^{n-m}.
\]

Where, the fourth equality follows from the proof of Theorem 2. \( \square \)

**Lemma 2[13, p. 17].** For \( s > -1 \) and \( t \in \mathbb{R} \), let

\[
I_{s,t}(z) = \int_D \frac{(1-|w|^2)^s}{|1-z \bar{w}|^{2+s+t}} dA(w), \quad z \in D
\]

then we have

1) \( I_{s,t}(z) \) is bounded in \( z \) if \( t < 0 \);
2) \( I_{s,t}(z) \sim -\log(1-|z|^2) \) as \( |z| \to 1^- \) if \( t = 0 \);
3) \( I_{s,t}(z) \sim (1-|z|^2)^{-t} \) as \( |z| \to 1^- \) if \( t > 0 \);

**Theorem 4.** For each \( q > 0 \), the operator \( T_q \) maps \( L^\infty(D) \) boundedly into \( B_q \).

**proof.** For every \( g \) in \( L^\infty(D) \),

\[
T_q g(z) = q \int_D \frac{g(w)}{(1-z \bar{w})^{1+q}} dA(w).
\]

\[
\frac{d}{dz}(T_q g(z)) = q(q+1) \int_D \frac{\bar{w} g(w)}{(1-z \bar{w})^{2+q}} dA(w).
\]
By Lemma 2,

\[ \left| \frac{d}{dz} (T_q g(z)) \right| \leq q(q + 1)\|g\|_\infty \int_D \frac{dA(w)}{|1 - z\overline{w}|^{2+q}} \]

\[ \leq C\|g\|_\infty (1 - |z|^2)^{-q} \]

for some constant $C > 0$. Since

\[ |T_q g(0)| \leq q \int_D g(w) dA(w) \leq q \|g\|_\infty, \]

we obtain the following desired result

\[ \|T_q g\|_q \leq (C + q)\|g\|_\infty. \]

\[\square\]

REFERENCES


DEPARTMENT OF MATHEMATICS, KONYANG UNIVERSITY, NONSAN 320-800, KOREA.

E-mail address: ksc @ kytis.konyang.ac.kr

DEPARTMENT OF MATHEMATICS, KONYANG UNIVERSITY, NONSAN 320-800, KOREA.

E-mail address: gtyang @ kytis.konyang.ac.kr