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MULTIPLE POSITIVE SOLUTIONS FOR PSEUDO-LAPLACIAN EQUATION WITH CRITICAL EXPONENTS

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ABSTRACT. This paper is concerned with the existence of mutiple positive solution of

$$-\Delta_p u = Q(x)|u|^{p^*-2}u + \lambda|u|^{p-2}u, \ x \in \Omega, u \in W_0^{1,p}(\Omega)$$

with Dirichlet boundary condition.

1. Introduction

We are concerned with the existence of multiple positive solutions of the following quasi-linear elliptic equation

$$\begin{aligned} -\Delta_p u &= Q(x)|u|^{p^*-2}u + \lambda|u|^{p-2}u \ \text{ on } \ \Omega, \quad u \in W_0^{1,p}(\Omega) \\ u &> 0 \quad \text{in } \ \Omega \\ u &= 0 \quad \text{on } \ \partial\Omega \end{aligned} \tag{1}$$

where Δ_p is called p-Laplace operator,defined by $\Delta_p u = \nabla(|\nabla u|^{p-2} \nabla u)$, Ω is a smooth bounded domain of $R^N (N \geq 4), N \geq p^2 > 1, p^* = \frac{pN}{N-p}, \lambda \in (0, \lambda_1)$ (λ_1 is the first eigenvalue of $-\Delta_p$ with zero Dirichlet boundary condition) and $Q(x) \in C(\bar{\Omega})$ satisfies the following condition:

Condition(Q). *$Q(x) \geq 0$ in Ω and there exist points $a^1, a^2, \cdots, a^k \in \Omega$ such that $Q(a^j)$ are strict local maximums satisfying*

$$Q(a^j) = Q_M = \underset{\bar{\Omega}}{Max} Q(x) > 0,$$

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$$Q(x) - Q(a^j) = \begin{cases} O(|x - a^j|^p) & if\, N = p^2, \\ o(|x - a^j|^p) & if\, N > p^2 > 1 \end{cases}$$

for x near a^j, $j = 1, 2, ..., k$.

Our main result in this paper is the following:

Main thereom. *Suppose that condition (Q) holds. Then there exists a positive number $\lambda_0 \in (0, \lambda_1)$ such that, for $\lambda \in (0, \lambda_0]$, problem(1) has at least k positive solutions.*

The existence of at least one positive solution of (1) has been established for the special case that $Q(x)$ is a constant by [3] and for $Q(x)$ satisfying condition(Q) at a point $a \in \bar{\Omega}$, by Escobar [7] for $p = 2$. This aim of this paper is to show the effect of the shape of the graph of $Q(x)$ on the existence and multiplicity of positive solutions. Our solutions are obtained as local minimum points of the functional

$$I_\lambda(u) = \frac{1}{p} \int_\Omega | \bigtriangledown u|^p - \lambda |u|^p - \frac{1}{p^*} \int_\Omega Q(x)|u|^{p^*}, \tag{2}$$

for $\lambda \in (0, \lambda_1)$, constrained to suitably constructed closed subsets of $W_0^{1,p}(\Omega)$. Our positive solutions will correspond to critical values in $(0,\ S^{N/p}/NQ_M^{(N-p)/p})$. Our results seem to suggest that the geometry of the graph of $Q(x)$ has a similar effect to the geometry of Ω, on the existence and multiplicity of both kinds of solutions.

2. Preliminary Results

Let $\|.\|$ denote the norm of $W_0^{1,p}(\Omega)$, $\|u\| = \int |\bigtriangledown u|^p$ for all $u \in W_0^{1,p}(\Omega)$. Here all integrals are Lebesgue integrals over Ω unless otherwise stated. Let $g : W_0^{1,p}(\Omega) \to R^N$ be defined by

$$g(u) = \frac{\int x|u|^{p^*}}{\int |u|^{p^*}}. \tag{3}$$

For $r > 0, y \in R^N$, set $B_r(y) = \{x \in R^N, |x - y| < r\}$ and let $\bar{B}_r(y), S_r(y)$ denote the closure and the boundary of $B_r(y)$ respectively. By condition(Q) we may choose $l > 0$ small enough so that $B_{2l}(a^j) \subset \Omega$ are disjoint and $Q(x) < Q(a^j)$ for $x \in B_{2l}(a^j), x \neq a^j, j = 1, 2, \cdots, k$. For $\lambda > 0$, define

$$\Sigma_\lambda = \{u \in W_0^{1,p}(\Omega) : u \not\equiv 0, < I'_\lambda(u), u >= 0\} \tag{4}$$

where $I'_\lambda(u)$ denotes the Frechet derivative of $I_\lambda(u)$. For $\lambda > 0$ *and* $j = 1, 2, \cdots, k$, define

$$O^j_\lambda = \{u \in \Sigma_\lambda : g(u) \in B_l(a^j)\},$$
$$U^j_\lambda = \{u \in \Sigma_\lambda : g(u) \in S_l(a^j)\}. \tag{5}$$

None of O^j_λ, U^j_λ are empty, as can be easily verified. Define

$$m^j_\lambda = inf\{I_\lambda(u) : u \in O^j_\lambda\},$$
$$\bar{m}^j_\lambda = inf\{I_\lambda(u) : u \in U^j_\lambda\}. \tag{6}$$

$$S = \inf\{\int_{R^N} |\bigtriangledown u|^p : u \in W^{1,p}_0(R^N), \int_{R^N} |u|^{p^*} = 1\}. \tag{7}$$

In the following Lemmas, we establish estimates on $m^j_\lambda, \bar{m}^j_\lambda$, which are crucial to our construction of P.S.sequences in the subsets O^j_λ.

Lemma 1. *For $j = 1, \cdots, k$ and $\lambda > 0$, we have*

$$m^j_\lambda < \frac{S^{\frac{N}{p}}}{NQ_M^{\frac{N-p}{p}}}. \tag{8}$$

Proof. Let $x_0 \in R^N, \varepsilon > 0$ and

$$\frac{U_{\varepsilon,x_0} = C_N \varepsilon^{(N-p)/p}}{(\varepsilon^p + |x - x_0|^{p/(p-1)})^{(N-p)/p}}$$

with

$$C_N = \left(\frac{N(\frac{N-p}{p-1})^{p-1}}{Q_M}\right)^{\frac{N-p}{p^2}}.$$

It is easy to verify that U_{ε,x_0} satisfies

$$-\Delta_p u = Q_M |u|^{p^*-2} u \quad \text{in} \quad R^N.$$

Furthermore, it is well known[3] that the infimum in (7) is achieved by the functions $U_{\varepsilon,x_0}/\|U_{\varepsilon,x_0}\|^{p^*}_{L(R^N)}$. Let $0 < \rho < \frac{l}{2}, \rho$ fixed. Define a radial nonnegative function $\phi \in C^2_0(R^N)$ by

$$\phi(x) = \begin{cases} 1 & \text{for } x \in B_\rho(0) \\ 0 & \text{for } x \notin B_{2\rho}(0). \end{cases}$$

Set $u_{\varepsilon,a^j} = U_{\varepsilon,a^j}\phi(x-a^j)$ and we shall simply write u_ε for u_{ε,a^j} when there is no confusion. The following estimates can be established from similar estimates in [3]: As $\varepsilon \to 0$,

$$\int |\nabla u_\varepsilon|^p = K_1 + O(\varepsilon^{N-p}),\ K_1 = \|\nabla U_{1,0}\|^p_{L^p(\Omega)}, \tag{9}$$

$$\int |u_\varepsilon|^{p^*} = K_2 + O(\varepsilon^N),\ K_2 = \|U_{1,0}\|^{p^*}_{L^{p^*}(\Omega)}, \tag{10}$$

$$(\int |u_\varepsilon|^{p^*})^{\frac{p}{p^*}} = K_2^{\frac{N-p}{N}} + O(\varepsilon^{N-p}),\ K_2^{\frac{N-p}{N}} = \|U_{1,0}\|^p_{L^{p^*}}, \tag{11}$$

$$\int |u_\varepsilon|^p = \begin{cases} K_3\varepsilon^{(p^2-p)} + O(\varepsilon^{N-p}) & \text{if } N > p^2 > 1, \\ K_3\varepsilon^{(p^2-p)}|log\varepsilon| + O(\varepsilon^{(p^2-p)}) & \text{if } N = p^2. \end{cases} \tag{12}$$

where K_3 is a positive constant. For $\varepsilon > 0$, let $t_\varepsilon > 0$ be selected such that $v_\varepsilon \equiv t_\varepsilon u_\varepsilon \in \Sigma_\lambda$. That is

$$t_\varepsilon^{p^*-p} = \frac{\int(|\nabla u_\varepsilon|^p - \lambda|u_\varepsilon|^p)}{\int Q(x)|u_\varepsilon|^{p^*}}. \tag{13}$$

Notice that $v_\varepsilon \in O^j_\lambda$. This follows easily from the symmetry of u_ε about a^j, which implies that $g(v_\varepsilon) \in B_l(a^j)$. Hence (8) will follow if we show that

$$\sup_{t>0} I_\lambda(tu_\varepsilon) < \frac{S^{\frac{N}{p}}}{NQ_M^{(N-p)/p}}. \tag{14}$$

To establish (16) we set

$$h(t) = \frac{t^p}{p}\int(|\nabla u_\varepsilon|^p - \lambda|u_\varepsilon|^p) - \frac{t^{p^*}}{p^*}\int Q(x)u_\varepsilon^{p^*},$$

$h'(t) > 0$ for $t \in (0, t_\varepsilon), h'(t) < 0$ for $t > t_\varepsilon$, and $h'(t_\varepsilon) = 0$. Therefore,

$$\begin{aligned}
&\sup_{t>0} I_\lambda(tu_\varepsilon) = I_\lambda(t_\varepsilon u_\varepsilon) \\
&= \frac{t_\varepsilon{}^p}{p}\int(|\nabla u_\varepsilon|^p - \lambda|u_\varepsilon|^p) - \frac{t_\varepsilon{}^{p^*}}{p^*}\int Q(x)u_\varepsilon{}^{p^*} \\
&= t_\varepsilon{}^p(\frac{1}{p} - \frac{1}{p^*})\int(|\nabla u_\varepsilon|^p - \lambda|u_\varepsilon|^p) \\
&= \frac{t_\varepsilon{}^p}{N}\int_\Omega |\nabla u_\varepsilon|^p - \lambda|u_\varepsilon|^p \\
&= \frac{1}{N}[(K_1 + O(\varepsilon^{N-p})) - K_3\eta(\varepsilon)\lambda]t_\varepsilon{}^p,
\end{aligned}$$

where

$$\eta(\varepsilon) = \begin{cases} \varepsilon^{(p^2-p)} & \text{if } N > p^2 > 1, \\ \varepsilon^{(p^2-p)}|\log \varepsilon| & \text{if } N = p^2 \end{cases}.$$

Using condition(Q) we have

$$\begin{aligned} \int Q(x){u_\varepsilon}^{p^*} &= \int Q_M |u_\varepsilon|^{p^*} + \int (Q(x) - Q_M)|u_\varepsilon|^{p^*} \\ &= \begin{cases} Q_M K_2 + O(\varepsilon^N) + o(\varepsilon^p) & \text{if } N > p^2 > 1, \\ Q_M K_2 + O(\varepsilon^N) + O(\varepsilon^p) & \text{if } N = p^2. \end{cases} \end{aligned} \tag{15}$$

Thus

$${t_\varepsilon}^p = \Big(\frac{K_1 + O(\varepsilon^{N-p}) - \lambda K_3 \eta(\varepsilon)}{Q_M K_2 + O(\varepsilon^N) + \alpha(\varepsilon)} \Big)^{\frac{N-p}{p}} \tag{16}$$

where

$$\alpha(\varepsilon) = \begin{cases} o(\varepsilon^p) & \text{if } N > p^2 > 1, \\ O(\varepsilon^p) & \text{if } N = p^2. \end{cases}$$

Using (16) and $\| \nabla U_{1,0}\|^p_{L^p(\Omega)} = S\|U_{1,0}\|^p_{L^{p^*}(\Omega)}$, we have

$$\sup_{t>0} I_\lambda(tu_\varepsilon) = \frac{K_1^{\frac{N}{p}}}{NQ_M^{\frac{N-p}{p}}(K_2)^{\frac{N-p}{p}}}\beta(\varepsilon) = \frac{1}{N}\frac{S^{\frac{N}{p}}}{Q_M^{\frac{N-p}{p}}}\beta(\varepsilon),$$

where

$$\beta(\varepsilon) = \begin{cases} 1 - \frac{N}{p}\frac{K_3}{K_1}\lambda\varepsilon^p + o(\varepsilon^p) & \text{if } N > p^2 > 1, \\ 1 - p\frac{K_3}{K_1}\lambda\varepsilon^p|\log\varepsilon| + O(\varepsilon^p) & \text{if } N = p^2. \end{cases}$$

It then follows that $\beta(\varepsilon) < 1$ for sufficiently small ε, which implies (16) and Lemma 1 follows.

Lemma 2. *Assume that condition (Q) holds. Then there exist $\varepsilon > 0$ and λ_ε such that*

$$\bar{m}^j_\lambda > \frac{S^{\frac{N}{p}}}{NQ_M^{\frac{N-p}{p}}} + \varepsilon,$$

for $j = 1, 2, ..., k$ and $\lambda \in (0, \lambda_\varepsilon)$.

Proof. suppose to the contrary that we could find a sequence λ_n as $n \to \infty$, such that $\bar{m}^j_{\lambda_n} \to c \le S^{N/p}/NQ_M^{(N-p)/p}$. Consequently, there exists $u_n \in U^j_{\lambda_n}$ such that

$$\begin{gathered} I_{\lambda_n}(u_n) \to c, \\ \int |\nabla u_n|^p - \lambda_n |u_n|^p = \int Q(x)|u_n|^{p^*}. \end{gathered} \tag{17}$$

It then follows easily that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and $\lambda_n \int |u_n|^p \to 0$ as $n \to \infty$. By Hölder's and Sobolev's inequalities we can fix $\nu > 0$ such that

$$\int |\bigtriangledown u_n|^p, \int Q(x)|u_n|^{p^*} \geq \nu > 0$$

for all $n = 1, 2, \cdots$. Therefore,we may choose $t_n > 0$ so that $v_n = t_n u_n$ satisfies

$$\int |\bigtriangledown v_n|^p = \int Q_M |v_n|^{p^*},$$

and

$$t_n = \Big(\frac{\int Q(x)|u_n|^{p^*} + \lambda_n \int |u_n|^p}{\int Q_M |u_n|^{p^*}} \Big)^{\frac{N-p}{p^2}}$$

are bounded. Suppose $t_n \to t_0$ as $n \to \infty$. Then $t_0 \leq 1$ since $Q(x) \leq Q_M$ and $\lambda_n \int |u_n|^p \to 0$ as $n \to \infty$. In fact, $t_0 = 1$. This follows easily from

$$\begin{aligned}
\frac{S^{\frac{N}{p}}}{NQ_M^{(N-p)/p}} &\leq \lim_{n\to\infty} \frac{1}{N} \int |\bigtriangledown v_n|^p = \lim_{n\to\infty} \frac{t_n^p}{N} \int |\bigtriangledown u_n|^p \\
&= \lim_{n\to\infty} t_n^p \frac{1}{N} \int (|\bigtriangledown u_n|^p - \lambda_n |u_n|^p) \\
&= \lim_{n\to\infty} t_n^p I_{\lambda_n}(u_n) = t_0^p c \\
&\leq t_0^p \frac{S^{\frac{N}{p}}}{NQ_M^{(N-p)/p}}.
\end{aligned}$$

The inequalities above also show that

$$c = \frac{S^{\frac{N}{p}}}{NQ_M^{(N-p)/p}} \quad \text{and} \quad \lim_{n\to\infty} \int |\bigtriangledown v_n|^p = \frac{S^{\frac{N}{p}}}{Q_M^{(N-p)/p}}. \tag{18}$$

Set $w_n = v_n / (\int |v_n|^{p^*})^{1/p^*}$. It is easy to verify that

$$\int |\bigtriangledown w_n|^p \to S \quad \text{as} \quad n \to \infty.$$

That is,$\{w_n\}$ is a minimizing sequence for the problem

$$S = \inf \Big(\int |\bigtriangledown u|^p : u \in W_0^{1,p}(\Omega), \int |u|^{p^*} = 1 \Big).$$

We now use a result of P.L.Lions[9] to conclude that we can find a point $x_0 \in \bar{\Omega}$ and a subsequence, still denoted by $\{w_n\}$, such that

$$\lim_{n\to\infty} \int_\Omega v|w_n|^{p^*} = v(x_0) \tag{19}$$

for any $v \in C(\bar{\Omega})$. In particular, we have

$$g^i(u_n) = \frac{\int x_i|u_n|^{p^*}}{\int |u_n|^{p^*}} = \frac{\int x_i|w_n|^{p^*}}{\int |w_n|^{p^*}} \to (x_0)_i,$$

as $n \to \infty$, and since $g(u_n) \in S_l(a^j), x_0 \in S_l(a^j)$. Using (18) and (19) we also have

$$\begin{aligned}\lim_{n\to\infty} \int Q(x)|u_n|^p &= \lim_{n\to\infty} \int Q(x)|v_n|^{p^*} \\ &= \frac{Q(x_0)}{Q_M} \lim_{n\to\infty} \int Q_M|v_n|^{p^*} \\ &= \frac{Q(x_0)}{Q_M} \lim_{n\to\infty} \int |\nabla v_n|^p \\ &= \frac{Q(x_0)}{Q_M} \frac{S^N}{Q_M^{\frac{N-p}{p}}},\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} I_{\lambda_n}(u_n) &= \frac{1}{N} \lim_{n\to\infty} \int Q(x)|u_n|^{p^*} \\ &= \frac{1}{N} \lim_{n\to\infty} \int Q(x)|v_n|^{p^*} \\ &= \frac{Q(x_0)}{NQ_M} \lim_{n\to\infty} \int Q_M|v_n|^{p^*} \\ &= \frac{Q(x_0)}{Q_M} \frac{S^N}{NQ_M^{\frac{N-p}{p}}} < \frac{S^{\frac{N}{p}}}{NQ_M^{(N-p)/p}},\end{aligned}$$

contradicting (18). Hence Lemma 2 follows.

Lemma 3. *Assume that condition (Q) holds and $\lambda \in (0, \lambda_1)$. Then any sequence $\{u_n\} \subset \Sigma_\lambda$ satisfying*

$$I_\lambda(u_n) \to c < \frac{S^{\frac{N}{p}}}{NQ_M^{(N-p)/p}},$$
$$I'_\lambda(u_n) \to 0 \quad as \quad n \to \infty,$$

is relatively compact in $W_0^{1,p}(\Omega)$.

Proof. Since $\lambda \in (0, \lambda_1)$, it is easy to show that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and therefore we may assume that for some $u_0 \in W_0^{1,p}(\Omega)$,

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W_0^{1,p}(\Omega),$$

$$u_n \to u_0 \quad \text{a.e. in } \Omega,$$

$$\int |\nabla u_0|^p - \lambda |u_0|^p = \int Q(x)|u_0|^{p^*},$$

$$I_\lambda(u_0) = \frac{1}{N}\int Q(x)|u_0|^{p^*} > 0 \quad \text{if} \quad u_0 \neq 0.$$

Let $w_n = u_n - u_0$. By Brezis-Lieb Lemma[10], we have

$$\int Q(x)|u_n|^{p^*} = \int Q(x)|u_0|^{p^*} + \int Q(x)|w_n|^{p^*} + o(1),$$

$$\int |\nabla u_n|^p = \int |\nabla u_0|^p + \int |\nabla w_n|^p + o(1).$$

If $\|w_n\| \to 0$,we are done, so assume $\|w_n\| \to L > 0$. It follows from

$$\begin{aligned}
\int |\nabla w_n|^p &= \int |\nabla u_n|^p - \int |\nabla u_0|^p + o(1) \\
&= \lambda \int |u_n|^p + \int Q(x)|u_n|^{p^*} - \int |\nabla u_0|^p + o(1) \\
&= \lambda \int |u_n|^p + \int Q(x)|w_n|^{p^*} + \int Q(x)|u_0|^{p^*} - \int |\nabla u_0|^p + o(1) \\
&= \int Q(x)|w_n|^{p^*} + o(1),
\end{aligned}$$

that we can choose $t_n > 0$ so that

$$\int |\nabla t_n w_n|^p = \int Q_M |t_n w_n|^{p^*}.$$

In fact

$$t_n^{\frac{p^2}{N-p}} = \frac{\int Q(x)|w_n|^{p^*} + o(1)}{\int Q_M |w_n|^{p^*}},$$

and $t_n \to t_0 \le 1$. Therefore,

$$\begin{aligned}\frac{1}{N}\lim_{n\to\infty}\int |\nabla w_n|^p &\ge \frac{t_0^p}{N}\lim_{n\to\infty}\int |\nabla w_n|^p \\ &= \frac{1}{N}\lim_{n\to\infty}\int |\nabla t_n w_n|^p \\ &\ge \frac{S^{\frac{N}{p}}}{NQ_M^{\frac{N-p}{N}}},\end{aligned}$$

where the last inequality follows from the definition of S. We then have

$$\lim_{n\to\infty}\int |\nabla w_n|^p \ge \frac{S^{\frac{N}{p}}}{Q_M^{\frac{N-p}{p}}},$$

and therefore

$$\begin{aligned}\lim_{n\to\infty} I_\lambda(u_n) &= I_\lambda(u_0) + \lim_{n\to\infty} I_\lambda(w_n) \\ &= I_\lambda(u_0) + \frac{1}{N}\lim_{n\to\infty}\int |\nabla w_n|^p \\ &\ge I_\lambda(u_0) + \frac{S^{\frac{n}{p}}}{NQ_M^{\frac{N-p}{p}}} \\ &\ge \frac{S^{\frac{N}{p}}}{NQ_M^{\frac{N-p}{p}}},\end{aligned}$$

contradicting the hypothesis. Therefore $\|w_n\| \to L > 0$ is impossible, and Lemma 3 follows.

Lemma 4. *Assume that condition (Q) holds. Then there exists a $\lambda_0 \in (0, \lambda_1)$ and a sequence $\{u_n^j\} \subset O_\lambda^j$, for each $j = 1, 2, \cdots, k$, satisfying $u_n \ge 0$,*

$$I_\lambda(u_n^j) \to m_\lambda^j, \tag{20}$$

$$I'_\lambda(u_n^j) \to 0, \tag{21}$$

as $n \to \infty$, for $\lambda \in (0, \lambda_0]$.

Proof. We first notice that for $\lambda < \lambda_1$ there is a positive constant $\delta = (\lambda_1 - \lambda)$ such that $\|u\|_{L^{p^*}(\Omega)} \ge \delta > 0$ for all $u \in O_\lambda^j$. Therefore, $\bar{O}_\lambda^j = O_\lambda^j \cup U_\lambda^j$ and U_λ^j is the boundary of $\bar{O}_\lambda^j$ for $\lambda < \lambda_1$ and each $j = 1, 2, \cdots, k$. Using Lemma 1 and Lemma 2 we see that there exists $\lambda_0 \in (0, \lambda_1)$ such that

$$m_\lambda^j < \bar{m}_\lambda^j \tag{22}$$

for $\lambda \in (0, \lambda_0], j = 1, 2, \cdots, k$. It follows that

$$m_\lambda^j = \inf\{I_\lambda(u) : u \in \bar{O}_\lambda^j\}. \tag{23}$$

Fix $\lambda \in (0, \lambda_0]$ and let $\{u_n^j\} \subset \bar{O}_\lambda^j$ be a minimizing sequence for (22). By replacing u_n^j with $|u_n^j|$, if necessary, we may assume that $u_n^j \geq 0$. By applying Ekeland's variational principle[6] we construct a minimizing sequence $\{v_n\} \subset \bar{O}_\lambda^j$, for each $j = 1, 2, \cdots, k$, with the properties

$$\begin{aligned} &\text{(a)} \quad I_\lambda(v_n) \leq I_\lambda(u_n^j) < m_\lambda^j + \frac{1}{n}, \\ &\text{(b)} \quad \|v_n - u_n^j\| \leq \frac{1}{n}, \\ &\text{(c)} \quad I_\lambda(v_n) < I_\lambda(w) + \frac{1}{n}\|w - v_n\| \quad \text{for each} \quad w \neq v_n \quad \text{in} \quad \bar{O}_\lambda^j. \end{aligned} \tag{24}$$

Using (22) we may assume that $v_n \in O_\lambda^j$ for sufficiently large n. We may now employ the argument in [6] to construct,for each v_n, an $\varepsilon_n > 0$ and a functional $t^n(w)$ defined for $w \in W_0^{1,p}(\Omega)$, $\|w\| \leq \varepsilon_n$ such that $t^n(w)(v_n - w) \in O_\lambda^j$, and

$$\langle (t^n)'(0), v\rangle = \frac{p\int(|\nabla v_n|^{p-2}.\nabla v - \lambda|v_n|^{p-2}v) - p^*\int Q(x)|v_n|^{p^*-2}v_n v}{\int|\nabla v_n|^p - \lambda|v_n|^p - (p^*-1)\int Q(x)|v_n|^{p^*}}. \tag{25}$$

Choose $0 < \delta < \varepsilon_n$. Let $0 \neq u \in W_0^{1,p}(\Omega)$ and let $w_\delta = \frac{\delta u}{\|u\|}$. Fix n and let $z_\delta = t^n(w_\delta)(v_n - w_\delta)$. Since $z_\delta \in O_\lambda^j$ by the properties of $t^n(w_\delta)$,

$$I_\lambda(z_\delta) - I_\lambda(v_n) \geq -\frac{1}{n}\|z_\delta - v_n\|$$

follows from (24). The mean value theorem then gives

$$\langle I_\lambda'(v_n), z_\delta - v_n\rangle + o(\|z_\delta - v_n\|) \geq -\frac{1}{n}\|z_\delta - v_n\|.$$

Hence

$$\langle I'_\lambda(v_n), (v_n - w_\delta) + (t^n(w_\delta) - 1)(v_n - w_\delta) - v_n\rangle \quad \geq \quad -\frac{1}{n}\|z_\delta - v_n\| + o(\|z_\delta - v_n\|),$$

which implies that

$$\begin{aligned} &- \langle I'_\lambda(v_n), w_\delta\rangle + (t^n(w_\delta) - 1)\langle I'_\lambda(v_n), v_n - w_\delta\rangle \\ &\geq -\frac{1}{n}\|z_\delta - v_n\| + o(\|z_\delta - v_n\|). \end{aligned} \tag{26}$$

Since $t^n(w_\delta)(v_n - w_\delta) \in O^j_\lambda$, $\langle I'_\lambda(z_\delta), t^n(w_\delta)(v_n - w_\delta)\rangle = 0$. Thus it follows from (26) that

$$\begin{aligned} -\delta\langle I'_\lambda(v_n), \frac{u}{\|u\|}\rangle + \frac{t^n(w_\delta) - 1}{t^n(w_\delta)} \langle I'_\lambda(z_\delta), t^n(w_\delta)(u_n - w_\delta)\rangle \\ + (t^n(w_\delta) - 1)\langle I'_\lambda(v_n) - I'_\lambda(z_\delta), v_n - w_\delta\rangle \\ \geq -\frac{1}{n}\|z_\delta - v_n\| + o(\|z_\delta - v_n\|). \end{aligned}$$

Hence

$$\begin{aligned} \langle I'_\lambda(v_n), \frac{u}{\|u\|}\rangle \quad \leq \quad & \frac{1}{n}\frac{\|z_\delta - v_n\|}{\delta} + \frac{o(\|z_\delta - v_n\|)}{\delta} \\ & + \frac{(t^n(w_\delta) - 1)}{\delta}\langle I'_\lambda(v_n) - I'_\lambda(z_\delta), u_n - w_\delta\rangle. \end{aligned} \tag{27}$$

But $\|z_\delta - v_n\| \leq \delta + |t^n(w_\delta) - 1|C$,

$$\lim_{\delta\to 0} \frac{|t^n(w_\delta) - 1|}{\delta} \leq \|(t^n)'(0)\| \leq C$$

for some constant $C > 0$, independent of δ, as can be easily verified from (25). For fixed n, letting $\delta \to 0$ in (27), we obtain

$$\langle I'_\lambda(v_n), \frac{u}{\|u\|}\rangle \quad \leq \frac{C}{n},$$

which implies that $I'_\lambda(v_n) \to 0$, as $n \to \infty$, and by (24)(b) we conclude that $I'_\lambda(u^j_n) \to 0$ for $\lambda \in (0, \lambda_0)$. This completes the proof of Lemma 4.

3. Proof of Main Theorem

By combining Lemma 3 and Lemma 4, we see that there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0], j = 1, 2, \cdots, k$, we have a minimizing sequence $\{u_n^j\} \subset O_\lambda^j$ such that

$$\begin{aligned} &u_n^j \geq 0, \\ &I_\lambda(u_n^j) \to m_\lambda^j, \\ &I'_\lambda(u_n^j) \to 0, \\ &u_n^j \to u^j \quad \text{strongly in} \quad W_0^{1,p}(\Omega). \end{aligned}$$

it then follows that $u^j \not\equiv 0$ is a weak solution of (1) and $u^j \geq 0$. By standard regularity argument and the Vazquez maximum principle[11], we obtain $u^j(x) > 0$ in Ω, and since $g(u^j) \in B_l(a^j)$ and $B_l(a^j)$ are disjoint for $j = 1, 2, \cdots, k$. we conclude that $u^j, j = 1, \cdots, k$, are distinct positive solutions of (1). This completes the proof of main Theorem.

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