MULTIPLE POSITIVE SOLUTIONS FOR PSEUDO-LAPLACIAN EQUATION WITH CRITICAL EXPONENTS

Kwon Wook Kim

ABSTRACT. This paper is concerned with the existence of multiple positive solution of

$$- \Delta_p u = Q(x)|u|^{p^*-2}u + \lambda|u|^{p-2}u, \ x \in \Omega, \ u \in W^{1,p}_0(\Omega)$$

with Dirichlet boundary condition.

1. Introduction

We are concerned with the existence of multiple positive solutions of the following quasi-linear elliptic equation

$$- \Delta_p u = Q(x)|u|^{p^*-2}u + \lambda|u|^{p-2}u \ \text{on} \ \Omega, \ \ u \in W^{1,p}_0(\Omega)$$

$$u > 0 \ \text{in} \ \Omega$$

$$u = 0 \ \text{on} \ \partial\Omega$$

(1)

where $\Delta_p$ is called $p$-Laplace operator, defined by $\Delta_p \ u = \nabla(|\nabla u|^{p-2} \ \nabla u)$, $\Omega$ is a smooth bounded domain of $R^N \ (N \geq 4), N \geq p^2 > 1, p^* = \frac{pN}{N-p}, \lambda \in (0, \lambda_1)$ ($\lambda_1$ is the first eigenvalue of $-\Delta_p$ with zero Dirichlet boundary condition) and $Q(x) \in C(\bar{\Omega})$ satisfies the following condition:

Condition(Q). $Q(x) \geq 0$ in $\Omega$ and there exist points $a^1, a^2, \cdots, a^k \in \Omega$ such that $Q(a^j)$ are strict local maximums satisfying

$$Q(a^j) = Q_M = \max_{\Omega} Q(x) > 0,$$
\[ Q(x) - Q(a^j) = \begin{cases} O(|x - a^j|^p) & \text{if } N = p^2, \\ o(|x - a^j|^p) & \text{if } N > p^2 > 1 \end{cases} \]

for \( x \) near \( a^j \), \( j = 1, 2, \ldots, k \).

Our main result in this paper is the following:

**Main theorem.** Suppose that condition (Q) holds. Then there exists a positive number \( \lambda_0 \in (0, \lambda_1) \) such that, for \( \lambda \in (0, \lambda_0] \), problem (1) has at least \( k \) positive solutions.

The existence of at least one positive solution of (1) has been established for the special case that \( Q(x) \) is a constant by [3] and for \( Q(x) \) satisfying condition (Q) at a point \( a \in \bar{\Omega} \), by Escobar [7] for \( p = 2 \). This aim of this paper is to show the effect of the shape of the graph of \( Q(x) \) on the existence and multiplicity of positive solutions. Our solutions are obtained as local minimum points of the functional

\[ I_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \lambda |u|^p - \frac{1}{p^*} \int_\Omega Q(x)|u|^{p^*}, \quad (2) \]

for \( \lambda \in (0, \lambda_1) \), constrained to suitably constructed closed subsets of \( W^{1,p}_0(\Omega) \). Our positive solutions will correspond to critical values in \((0, S^{N/p}/NQ_M^{(N-p)/p})\). Our results seem to suggest that the geometry of the graph of \( Q(x) \) has a similar effect to the geometry of \( \Omega \), on the existence and multiplicity of both kinds of solutions.

### 2. Preliminary Results

Let \( \| \| \) denote the norm of \( W^{1,p}_0(\Omega) \), \( \| u \| = \int |\nabla u|^p \) for all \( u \in W^{1,p}_0(\Omega) \). Here all integrals are Lebesgue integrals over \( \Omega \) unless otherwise stated. Let \( g : W^{1,p}_0(\Omega) \to R^N \) be defined by

\[ g(u) = \frac{\int x|u|^{p^*}}{\int |u|^{p^*}}. \quad (3) \]

For \( r > 0, y \in R^N \), set \( B_r(y) = \{ x \in R^N, |x - y| < r \} \) and let \( \bar{B}_r(y), S_r(y) \) denote the closure and the boundary of \( B_r(y) \) respectively. By condition (Q) we may choose \( l > 0 \) small enough so that \( B_{2l}(a^j) \subset \Omega \) are disjoint and \( Q(x) < Q(a^j) \) for \( x \in B_{2l}(a^j), x \neq a^j, j = 1, 2, \ldots, k \). For \( \lambda > 0 \), define

\[ \Sigma_\lambda = \{ u \in W^{1,p}_0(\Omega) : u \neq 0, < I'_\lambda(u), u >= 0 \} \quad (4) \]
where $I'_{\lambda}(u)$ denotes the Frechet derivative of $I_{\lambda}(u)$. For $\lambda > 0$ and $j = 1, 2, \ldots, k$, define

$$O_{\lambda}^j = \{u \in \Sigma_{\lambda} : g(u) \in B_{t}(a^j)\},$$

$$U_{\lambda}^j = \{u \in \Sigma_{\lambda} : g(u) \in S_{t}(a^j)\}. \quad (5)$$

None of $O_{\lambda}^j, U_{\lambda}^j$ are empty, as can be easily verified. Define

$$m_{\lambda}^j = \inf \{I_{\lambda}(u) : u \in O_{\lambda}^j\},$$

$$\bar{m}_{\lambda}^j = \inf \{I_{\lambda}(u) : u \in U_{\lambda}^j\}. \quad (6)$$

$$S = \inf \{\int_{R^{N}} |\nabla u|^{p} : u \in W_{0}^{1,p}(R^{N}), \int_{R^{N}} |u|^{p^*} = 1\}. \quad (7)$$

In the following Lemmas, we establish estimates on $m_{\lambda}^j, \bar{m}_{\lambda}^j$, which are crucial to our construction of P.S. sequences in the subsets $O_{\lambda}^j$.

**Lemma 1.** For $j = 1, \ldots, k$ and $\lambda > 0$, we have

$$m_{\lambda}^j < \frac{S_{N}^{\frac{p}{N-p}}}{NQ_{M}^{\frac{p}{p-1}}}. \quad (8)$$

**Proof.** Let $x_0 \in R^{N}, \varepsilon > 0$ and

$$U_{\varepsilon,x_0} = C_{N}\varepsilon^{(N-p)/p} \frac{\varepsilon^{p} + |x - x_0|^{p}/(p-1)(N-p)/p}{(\varepsilon^{p} + |x - x_0|^{p}/(p-1)(N-p)/p)}$$

with

$$C_{N} = \left(\frac{N(N-p)^{p-1}}{(N-1)^{p}}\right)^{\frac{N-p}{p-1}}.$$ 

It is easy to verify that $U_{\varepsilon,x_0}$ satisfies

$$-\Delta_{p} u = Q_{M}|u|^{p^*-2}u \quad \text{in} \quad R^{N}.$$ 

Furthermore, it is well known[3] that the infimum in (7) is achieved by the functions $U_{\varepsilon,x_0}/\|U_{\varepsilon,x_0}\|_{L(R^{N})}^{p^*}$. Let $0 < \rho < \frac{1}{2}, \rho$ fixed. Define a radial nonnegative function $\phi \in C_{0}^{2}(R^{N})$ by

$$\phi(x) = \begin{cases} 
1 & \text{for } x \in B_{\rho}(0) \\
0 & \text{for } x \notin B_{2\rho}(0).
\end{cases}$$
Set \( u_{\epsilon,a^j} = U_{\epsilon,a^j} \phi(x - a^j) \) and we shall simply write \( u_\epsilon \) for \( u_{\epsilon,a^j} \) when there is no confusion. The following estimates can be established from similar estimates in [3]:

As \( \epsilon \to 0 \),

\[
\int |\nabla u_\epsilon|^p = K_1 + O(\epsilon^{N-p}), \quad K_1 = \| \nabla U_{1,0} \|_{L^p(\Omega)}^p, \tag{9}
\]

\[
\int |u_\epsilon|^{p^*} = K_2 + O(\epsilon^N), \quad K_2 = \| U_{1,0} \|_{L^{p^*}(\Omega)}^{p^*}, \tag{10}
\]

\[
\left( \int |u_\epsilon|^{p^*} \right)^{\frac{p}{p^*}} = K_2^{\frac{N-p}{N}} + O(\epsilon^{N-p}), \quad K_2^{\frac{N-p}{N}} = \| U_{1,0} \|_{L^{p^*}}^p, \tag{11}
\]

\[
\int |u_\epsilon|^p = \begin{cases} 
K_3 \epsilon^{(p^2-p)} + O(\epsilon^{N-p}) & \text{if } N > p^2 > 1, \\
K_3 \epsilon^{(p^2-p)|\log\epsilon|} + O(\epsilon^{(p^2-p)}) & \text{if } N = p^2.
\end{cases} \tag{12}
\]

where \( K_3 \) is a positive constant. For \( \epsilon > 0 \), let \( t_\epsilon > 0 \) be selected such that \( v_\epsilon \equiv t_\epsilon u_\epsilon \in \Sigma_\lambda \). That is

\[
t^{p^* - p}_\epsilon = \frac{\int (|\nabla u_\epsilon|^p - \lambda |u_\epsilon|^p)}{\int Q(x)|u_\epsilon|^{p^*}}. \tag{13}
\]

Notice that \( v_\epsilon \in O^{\lambda}_\Theta \). This follows easily from the symmetry of \( u_\epsilon \) about \( a^j \), which implies that \( g(v_\epsilon) \in B_1(a^j) \). Hence (8) will follow if we show that

\[
\sup_{t > 0} I_\lambda(tu_\epsilon) < \frac{S^N_p}{NQ_{\bar{M}}^{(N-p)/p}}. \tag{14}
\]

To establish (16) we set

\[
h(t) = \frac{t^p}{p} \left( \int |\nabla u_\epsilon|^p - \lambda |u_\epsilon|^p \right) - \frac{t^{p^*}}{p^*} \int Q(x)u_\epsilon^{p^*},
\]

\( h'(t) > 0 \) for \( t \in (0,t_\epsilon) \), \( h'(t) < 0 \) for \( t > t_\epsilon \), and \( h'(t_\epsilon) = 0 \). Therefore,

\[
\sup_{t > 0} I_\lambda(tu_\epsilon) = I_\lambda(t_\epsilon u_\epsilon)
\]

\[
= \frac{t_\epsilon^p}{p} \left( \int |\nabla u_\epsilon|^p - \lambda |u_\epsilon|^p \right) - \frac{t_\epsilon^{p^*}}{p^*} \int Q(x)u_\epsilon^{p^*}
\]

\[
= t_\epsilon^p \left( \frac{1}{p} - \frac{1}{p^*} \right) \int |\nabla u_\epsilon|^p - \lambda |u_\epsilon|^p
\]

\[
= \frac{t_\epsilon^p}{N} \int_\Omega |\nabla u_\epsilon|^p - \lambda |u_\epsilon|^p
\]

\[
= \frac{1}{N} \left( (K_1 + O(\epsilon^{N-p})) - K_2 \eta(\epsilon) \lambda \right) t_\epsilon^p,
\]
where
\[
\eta(\varepsilon) = \begin{cases} 
\varepsilon^{(p^2-p)} & \text{if } N > p^2 > 1, \\
\varepsilon^{(p^2-p)}|\log\varepsilon| & \text{if } N = p^2.
\end{cases}
\]

Using condition \((Q)\) we have
\[
\int Q(x) u_{\varepsilon}^{p^*} = \int Q_M |u_{\varepsilon}|^{p^*} + \int (Q(x) - Q_M) |u_{\varepsilon}|^{p^*}
= \begin{cases} 
Q_M K_2 + O(\varepsilon^{N}) + o(\varepsilon^{p}) & \text{if } N > p^2 > 1, \\
Q_M K_2 + O(\varepsilon^{N}) + O(\varepsilon^{p}) & \text{if } N = p^2.
\end{cases}
\]

Thus
\[
t_{\varepsilon}^{p} = \left( \frac{K_1 + O(\varepsilon^{N-p}) - \lambda K_3 \eta(\varepsilon)}{Q_M K_2 + O(\varepsilon^{N}) + \alpha(\varepsilon)} \right)^{\frac{N-p}{p}}
\]  
(16)

where
\[
\alpha(\varepsilon) = \begin{cases} 
o(\varepsilon^{p}) & \text{if } N > p^2 > 1, \\
o(\varepsilon^{p}) & \text{if } N = p^2.
\end{cases}
\]

Using (16) and \( \| \nabla U_{1,0} \|_{L^p(\Omega)}^{p} = S \| U_{1,0} \|_{L^p(\Omega)}^{p} \), we have
\[
\sup_{t > 0} I_{\lambda\varepsilon}(tu_{\varepsilon}) = \frac{K_1^{N}}{NQ_{M}^{N-p} (K_2)^{N-p}} \beta(\varepsilon) = \frac{1}{N} S_{\lambda}^{N-p} \beta(\varepsilon),
\]

where
\[
\beta(\varepsilon) = \begin{cases} 
1 - K_1^{N-p} \lambda \varepsilon^{p} + o(\varepsilon^{p}) & \text{if } N > p^2 > 1, \\
1 - K_1^{N-p} \lambda \varepsilon^{p}|\log\varepsilon| + O(\varepsilon^{p}) & \text{if } N = p^2.
\end{cases}
\]

It then follows that \( \beta(\varepsilon) < 1 \) for sufficiently small \( \varepsilon \), which implies (16) and Lemma 1 follows.

**Lemma 2.** Assume that condition \((Q)\) holds. Then there exist \( \varepsilon > 0 \) and \( \lambda_{\varepsilon} \) such that
\[
\bar{m}_{\lambda}^{j} > \frac{S_{\lambda}^{N}}{NQ_{M}^{N-p}} + \varepsilon,
\]
for \( j = 1, 2, \ldots, k \) and \( \lambda \in (0, \lambda_{\varepsilon}) \).

**Proof.** Suppose to the contrary that we could find a sequence \( \lambda_{n} \) as \( n \to \infty \), such that \( \bar{m}_{\lambda_{n}}^{j} \to c \leq S^{N/p}/NQ_{M}^{(N-p)/p} \). Consequently, there exists \( u_{n} \in U_{\lambda_{n}}^{j} \) such that
\[
I_{\lambda_{n}}(u_{n}) \to c,
\]
\[
\int |\nabla u_{n}|^{p} - \lambda_{n}|u_{n}|^{p} = \int Q(x)|u_{n}|^{p^*}.
\]  
(17)
It then follows easily that \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \) and \( \lambda_n \int |u_n|^p \to 0 \) as \( n \to \infty \). By Hölder’s and Sobolev’s inequalities we can fix \( \nu > 0 \) such that

\[
\int |\nabla u_n|^p, \int Q(x)|u_n|^{p^*} \geq \nu > 0
\]

for all \( n = 1, 2, \cdots \). Therefore, we may choose \( t_n > 0 \) so that \( v_n = t_n u_n \) satisfies

\[
\int |\nabla v_n|^p = \int Q_M |v_n|^{p^*},
\]

and

\[
t_n = \left( \frac{\int Q(x)|u_n|^{p^*} + \lambda_n \int |u_n|^p}{\int Q_M |u_n|^{p^*}} \right)^{\frac{N-p}{p^*}}
\]

are bounded. Suppose \( t_n \to t_0 \) as \( n \to \infty \). Then \( t_0 \leq 1 \) since \( Q(x) \leq Q_M \) and \( \lambda_n \int |u_n|^p \to 0 \) as \( n \to \infty \). In fact, \( t_0 = 1 \). This follows easily from

\[
\frac{S^N_p}{NQ^{(N-p)/p}_M} \leq \lim_{n \to \infty} \frac{1}{N} \int |\nabla v_n|^p = \lim_{n \to \infty} \frac{t_n^p}{N} \int |\nabla u_n|^p = \lim_{n \to \infty} t_n^p \left( |\nabla u_n|^p - \lambda_n |u_n|^p \right) = \lim_{n \to \infty} t_n^p I_{\lambda_n}(u_n) = t_0^p c
\]

\[
\leq t_0^p \frac{S^N_p}{NQ^{(N-p)/p}_M}.
\]

The inequalities above also show that

\[
c = \frac{S^N_p}{NQ^{(N-p)/p}_M} \quad \text{and} \quad \lim_{n \to \infty} \int |\nabla v_n|^p = \frac{S^N_p}{Q^{(N-p)/p}_M}.
\]

Set \( w_n = v_n/(\int |v_n|^{p^*})^{1/p^*} \). It is easy to verify that

\[
\int |\nabla w_n|^p \to S \quad \text{as} \quad n \to \infty.
\]

That is, \( \{w_n\} \) is a minimizing sequence for the problem

\[
S = \inf \left\{ \int |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int |u|^{p^*} = 1 \right\}.
\]
We now use a result of P.L. Lions\cite{9} to conclude that we can find a point \(x_0 \in \bar{\Omega}\) and a subsequence, still denoted by \(\{w_n\}\), such that

\[
\lim_{n \to \infty} \int_{\Omega} v|w_n|^{p^*} = v(x_0)
\]

for any \(v \in C(\bar{\Omega})\). In particular, we have

\[
g^i(u_n) = \frac{\int x_i|u_n|^{p^*}}{\int |u_n|^{p^*}} = \frac{\int x_i|w_n|^{p^*}}{\int |w_n|^{p^*}} \to (x_0)_i,
\]

as \(n \to \infty\), and since \(g(u_n) \in S_1(a^3), x_0 \in S_1(a^3)\). Using (18) and (19) we also have

\[
\lim_{n \to \infty} \int Q(x)|u_n|^p = \lim_{n \to \infty} \int Q(x)|v_n|^{p^*}
\]

\[
= \frac{Q(x_0)}{Q_M} \lim_{n \to \infty} \int Q_M|v_n|^{p^*}
\]

\[
= \frac{Q(x_0)}{Q_M} \lim_{n \to \infty} \int |\nabla v_n|^p
\]

\[
= \frac{Q(x_0)}{Q_M} \frac{S^N}{Q_M^{\frac{N-p}{p}}},
\]

\[
\lim_{n \to \infty} I_{\lambda_n}(u_n) = \frac{1}{N} \lim_{n \to \infty} \int Q(x)|u_n|^{p^*}
\]

\[
= \frac{1}{N} \lim_{n \to \infty} \int Q(x)|v_n|^{p^*}
\]

\[
= \frac{Q(x_0)}{NQ_M} \lim_{n \to \infty} \int Q_M|v_n|^{p^*}
\]

\[
= \frac{Q(x_0)}{Q_M} \frac{S^N}{NQ_M^{\frac{N-p}{p}}} \leq \frac{S^N}{NQ_M^{\frac{N-p}{p}}},
\]

contradicting (18). Hence Lemma 2 follows.

**Lemma 3.** Assume that condition (Q) holds and \(\lambda \in (0, \lambda_1)\). Then any sequence \(\{u_n\} \subset \Sigma_{\lambda}\) satisfying

\[
I_\lambda(u_n) \to c < \frac{S^N}{NQ_M^{(N-p)/p}},
\]

\[
I'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty,
\]
is relatively compact in $W^{1,p}_0(\Omega)$.

**Proof.** Since $\lambda \in (0, \lambda_1)$, it is easy to show that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$ and therefore we may assume that for some $u_0 \in W^{1,p}_0(\Omega)$,

$$u_n \to u_0 \quad \text{weakly in } W^{1,p}_0(\Omega),$$

$$u_n \to u_0 \quad \text{a.e. in } \Omega,$$

$$\int |\nabla u_0|^p - \lambda |u_0|^p = \int Q(x)|u_0|^{p^*},$$

$$I_{\lambda}(u_0) = \frac{1}{N} \int Q(x)|u_0|^{p^*} > 0 \quad \text{if } u_0 \neq 0.$$

Let $w_n = u_n - u_0$. By Brezis-Lieb Lemma[10], we have

$$\int Q(x)|u_n|^{p^*} = \int Q(x)|u_0|^{p^*} + \int Q(x)|w_n|^{p^*} + o(1),$$

$$\int |\nabla u_n|^p = \int |\nabla u_0|^p + \int |\nabla w_n|^p + o(1).$$

If $\|w_n\| \to 0$, we are done, so assume $\|w_n\| \to L > 0$. It follows from

$$\int |\nabla w_n|^p = \int |\nabla u_n|^p - \int |\nabla u_0|^p + o(1)$$

$$= \lambda \int |u_n|^p + \int Q(x)|u_n|^{p^*} - \int |\nabla u_0|^p + o(1)$$

$$= \lambda \int |u_n|^p + \int Q(x)|w_n|^{p^*} + \int Q(x)|u_0|^{p^*} - \int |\nabla u_0|^p + o(1)$$

$$= \int Q(x)|w_n|^{p^*} + o(1),$$

that we can choose $t_n > 0$ so that

$$\int |\nabla t_n w_n|^p = \int Q_M |t_n w_n|^{p^*}.$$
and \( t_n \to t_0 \leq 1 \). Therefore,

\[
\frac{1}{N} \lim_{n \to \infty} \int | \nabla w_n |^p \geq \frac{t_0^p}{N} \lim_{n \to \infty} \int | \nabla w_n |^p \\
= \frac{1}{N} \lim_{n \to \infty} \int | \nabla t_n w_n |^p \\
\geq \frac{S_p^N}{N Q_M^{\frac{N-p}{p}}},
\]

where the last inequality follows from the definition of \( S \). We then have

\[
\lim_{n \to \infty} \int | \nabla w_n |^p \geq \frac{S_p^N}{Q_M^p},
\]

and therefore

\[
\lim_{n \to \infty} I_\lambda(u_n) = I_\lambda(u_0) + \lim_{n \to \infty} I_\lambda(w_n) \\
= I_\lambda(u_0) + \frac{1}{N} \lim_{n \to \infty} \int | \nabla w_n |^p \\
\geq I_\lambda(u_0) + \frac{S_p^N}{Q_M^{\frac{N-p}{p}}} \\
\geq \frac{S_p^N}{N Q_M^{\frac{N-p}{p}}},
\]

contradicting the hypothesis. Therefore \( \|w_n\| \to L > 0 \) is impossible, and Lemma 3 follows.

**Lemma 4.** Assume that condition (Q) holds. Then there exists a \( \lambda_0 \in (0, \lambda_1) \) and a sequence \( \{u_n^j\} \subset O_\lambda^j \), for each \( j = 1, 2, \ldots, k \), satisfying \( u_n \geq 0 \),

\[
I_\lambda(u_n^j) \to m_\lambda^j, \tag{20}
\]

\[
I'_\lambda(u_n^j) \to 0, \tag{21}
\]

as \( n \to \infty \), for \( \lambda \in (0, \lambda_0] \).

**Proof.** We first notice that for \( \lambda < \lambda_1 \) there is a positive constant \( \delta = (\lambda_1 - \lambda) \) such that \( \|u\|_{L^p(\Omega)} \geq \delta > 0 \) for all \( u \in O_\lambda^j \). Therefore, \( \partial O_\lambda^j = O_\lambda^j \cup U_\lambda^j \) and \( U_\lambda^j \) is
the boundary of $\bar{O}_\lambda^j$ for $\lambda < \lambda_1$ and each $j = 1, 2, \cdots, k$. Using Lemma 1 and Lemma 2 we see that there exists $\lambda_0 \in (0, \lambda_1)$ such that

$$m_\lambda^j < \tilde{m}_\lambda^j$$

(22)

for $\lambda \in (0, \lambda_0], j = 1, 2, \cdots, k$. It follows that

$$m_\lambda^j = \inf\{I_\lambda(u) : u \in \bar{O}_\lambda^j\}.$$  

(23)

Fix $\lambda \in (0, \lambda_0]$ and let $\{u_n^j\} \subset \bar{O}_\lambda^j$ be a minimizing sequence for (22). By replacing $u_n^j$ with $|u_n^j|$, if necessary, we may assume that $u_n^j \geq 0$. By applying Ekeland’s variational principle[6] we construct a minimizing sequence $\{v_n\} \subset \bar{O}_\lambda^j$, for each $j = 1, 2, \cdots, k$, with the properties

(a) $I_\lambda(v_n) \leq I_\lambda(u_n^j) < m_\lambda^j + \frac{1}{n},$

(b) $\|v_n - u_n^j\| \leq \frac{1}{n},$

(24)

(c) $I_\lambda(v_n) < I_\lambda(w) + \frac{1}{n}\|w - v_n\|$ for each $w \neq v_n$ in $\bar{O}_\lambda^j$.

Using (22) we may assume that $v_n \in O_\lambda^j$ for sufficiently large $n$. We may now employ the argument in [6] to construct, for each $v_n$, an $\varepsilon_n > 0$ and a functional $t^n(w)$ defined for $w \in W_0^{1,p}(\Omega), \|w\| \leq \varepsilon_n$ such that $t^n(w)(v_n - w) \in O_\lambda^j$, and

$$\langle (t^n)'(0), v \rangle = \frac{p \int (|\nabla v_n|^{p-2} \cdot \nabla v - \lambda |v_n|^{p-2}v) - p^* \int Q(x)|v_n|^{p^*-2}v_nv}{\int |\nabla v_n|^{p-\lambda} |v_n|^p - (p^* - 1) \int Q(x)|v_n|^{p^*}}.$$  

(25)

Choose $0 < \delta < \varepsilon_n$. Let $0 \neq u \in W_0^{1,p}(\Omega)$ and let $w_\delta = \frac{\delta u}{\|u\|}$. Fix $n$ and let $z_\delta = t^n(w_\delta)(v_n - w_\delta)$. Since $z_\delta \in O_\lambda^j$ by the properties of $t^n(w_\delta)$,

$$I_\lambda(z_\delta) - I_\lambda(v_n) \geq -\frac{1}{n}\|z_\delta - v_n\|$$

follows from (24). The mean value theorem then gives

$$\langle I_\lambda'(v_n), z_\delta - v_n \rangle + o(\|z_\delta - v_n\|) \geq -\frac{1}{n}\|z_\delta - v_n\|.$$
Hence

\[ \langle I'_\lambda(v_n), (v_n - w_\delta) + (t^n(w_\delta) - 1)(v_n - w_\delta) - v_n \rangle \geq -\frac{1}{n} \|z_\delta - v_n\| + o(\|z_\delta - v_n\|),\]

which implies that

\[
-\langle I'_\lambda(v_n), w_\delta \rangle + (t^n(w_\delta) - 1)\langle I'_\lambda(v_n), v_n - w_\delta \rangle \\
\geq -\frac{1}{n} \|z_\delta - v_n\| + o(\|z_\delta - v_n\|). \tag{26}
\]

Since \( t^n(w_\delta)(v_n - w_\delta) \in O^J, \quad \langle I'_\lambda(z_\delta), t^n(w_\delta)(v_n - w_\delta) \rangle = 0 \). Thus it follows from (26) that

\[
-\delta \langle I'_\lambda(v_n), \frac{u}{\|u\|} \rangle + \frac{t^n(w_\delta) - 1}{t^n(w_\delta)} \langle I'_\lambda(z_\delta), t^n(w_\delta)(u_n - w_\delta) \rangle \\
+ (t^n(w_\delta) - 1)\langle I'_\lambda(v_n) - I'_\lambda(z_\delta), v_n - w_\delta \rangle \\
\geq -\frac{1}{n} \|z_\delta - v_n\| + o(\|z_\delta - v_n\|).
\]

Hence

\[
\langle I'_\lambda(v_n), \frac{u}{\|u\|} \rangle \leq \frac{1}{n} \|z_\delta - v_n\| + \frac{o(\|z_\delta - v_n\|)}{\delta} \\
+ \frac{(t^n(w_\delta) - 1)}{\delta} \langle I'_\lambda(v_n) - I'_\lambda(z_\delta), u_n - w_\delta \rangle. \tag{27}
\]

But \( \|z_\delta - v_n\| \leq \delta + |t^n(w_\delta) - 1|C, \)

\[
\lim_{\delta \to 0} \frac{|t^n(w_\delta) - 1|}{\delta} \leq \|(t^n)'(0)\| \leq C
\]

for some constant \( C > 0 \), independent of \( \delta \), as can be easily verified from (25). For fixed \( n \), letting \( \delta \to 0 \) in (27), we obtain

\[
\langle I'_\lambda(v_n), \frac{u}{\|u\|} \rangle \leq \frac{C}{n},
\]

which implies that \( I'_\lambda(v_n) \to 0 \), as \( n \to \infty \), and by (24)(b) we conclude that \( I'_\lambda(w_n^\lambda) \to 0 \) for \( \lambda \in (0, \lambda_0) \). This completes the proof of Lemma 4.

3. Proof of Main Theorem
By combining Lemma 3 and Lemma 4, we see that there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0], j = 1, 2, \cdots, k$, we have a minimizing sequence $\{u_n^j\} \subset O_\lambda^j$ such that

\[
\begin{align*}
\quad u_n^j &\geq 0, \\
I_\lambda(u_n^j) &\to m_\lambda^j, \\
I_\lambda'(u_n^j) &\to 0, \\
I_\lambda^j &\to u^j \text{ strongly in } W_0^{1,p}(\Omega).
\end{align*}
\]

it then follows that $u^j \neq 0$ is a weak solution of (1) and $u^j \geq 0$. By standard regularity argument and the Vazquez maximum principle[11], we obtain $u^j(x) > 0$ in $\Omega$, and since $g(u^j) \in B_l(a^j)$ and $B_l(a^j)$ are disjoint for $j = 1, 2, \cdots, k$. we conclude that $u^j, j = 1, \cdots, k$, are distinct positive solutions of (1). This completes the proof of main Theorem.

REFERENCES


Department of Mathematics education, Sunchon national University, Sunchon, Cheonnam 540-742, Korea.