J. Korea Soc. Math. Ed. Ser. B: Pure Appl. Math.
5(1998), no. 1, 45–54

UTI WARPED PRODUCT SPACE-TIME AND CAUSAL BOUNDARY OF UTI SPACE-TIME

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ABSTRACT. We study the space-times that have a unique terminal indecomposable past set or a unique terminal indecomposable future set and examine their causal boundary, and we investigate some conditions for the warped product space-times of the form $(a, b) \times_f F$ to have a unique terminal indecomposable past set or a unique terminal indecomposable future set.

1. Introduction and Preliminaries

A space-time represents a time-oriented Lorentzian manifold M. The most important subject for space-time is to study singularities. In the developments concerning the theory of occurrence of singularities in a space-time, the causal structure of the space-time has been extensively discussed and various causality conditions have been considered in many physical and mathematical situations and characterized in terms of various methods by many authors. If we regard singularities simply as points of a boundary of a space-time manifold, one can ask how such a boundary should be constructed. In order to have a better description of space-time singularities, one would like to construct an enlarged topological space $\bar{M}$ interpreted as the space-time manifold M with some singular boundary attached. $\bar{M}$ has a unified structure incorporating singular as well as nonsingular points. Several attempts have been made to construct a boundary to a space-time and the boundary structures are examined ([2], [3], [6], [9], [11], [12] *etc.*). Among various constructions, the causal boundary construction given by Geroch, Kronheimer and Penrose [3] is probably the most interesting one even though the enlarged space $\bar{M}$ is constructed

Received by the editors March 9, 1998 and, in revised form April 22, 1998.
1991 *Mathematics Subject Classifications.* Primary 53C50, 83C75.
Key words and phrases. Warped product space-time, (proper, terminal) indecomposable set, UTI space-time, causal boundary.
This work was supported by the Yeungnam University Research Grants in 1997.

in an implicit way to be Hausdorff topological space. To examine the causal boundary construction, one has to investigate indecomposable sets and to do it one has to study the future (past) inextendible nonspacelike curves.

In this paper, we study space-times that have a unique terminal indecomposable past set or a unique terminal indecomposable future set and examine their causal boundary, and we investigate some conditions for the warped product space-times of the form $(a,b) \times_f F$ to have a unique terminal indecomposable past set or a unique terminal indecomposable future set.

The standard space-time models of the universe are warped products, for example, the exterior Schwarzschild space-time and Robertson-Walker space-time etc. Robertson-Walker space-times include Minkowski space and Einstein static universe space-time. Several properties of Lorentzian warped products have been studied and examined geometrically ([1],[7]).

We first recall the concepts and notations of Lorentzian warped products. Suppose that (B, g_B) is a Lorentzian manifold and (F, g_F) is a Riemannian manifold and let f be a positive smooth function on B. The *Lorentzian warped product* $B \times_f F$ is the product manifold $B \times F$ furnished with the Lorentzian metric $g = g_B \oplus f g_F$ which is defined for tangent vectors v, w to $B \times_f F$ at $p = (p_1, p_2)$ by

$$g(v,w) = g_B(d\pi(v), d\pi(w)) + f(p_1) g_F(d\sigma(v), d\sigma(w)).$$

where π and σ are the projections of $B \times_f F$ onto B and F respectively. In the case that B is a space-time, the warped product $B \times_f F$ is also a space-time, it is called a *warped product space-time.*

Now, we state some concepts and notations which are used for our subsequent works. As usual, the chronological (causal) relation is written as $\ll$ $(<)$, and $I^+(A)$ $(I^-(A))$, $J^+(A)$ $(J^-(A))$ shall denote the *chronological future* (*past*), *causal future* (*past*) of a subset A in a space-time M. For a single point p, we abbreviate $I^+(\{p\})$ by $I^+(p)$ and similarly for I^-, J^+ and J^-. A space-time M is *distinguishing* if $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$. M is *strongly causal* if for every point p in M and any neighborhood U of p there is a neighborhood V of p contained in U such that no nonspacelike curve intersects V more than once. M is *stably causal* if it admits a global time function, *i.e.*, a C^0 function from M to $\mathbb{R}$ which is strictly increasing along each future-directed nonspacelike curve. We say that M is *causally continuous* if it is distinguishing and $I^+(p) \subseteq I^+(q)$ iff $I^-(q) \subseteq I^-(p)$ for all $p, q \in M$. M is *causally simple* if it is distinguishing and $J^+(p)$ and $J^-(p)$ are closed for all $p \in M$. A strongly causal space-time M is *globally hyperbolic* if for each pair of points p, q in M, the set $J^+(p) \cap J^-(p)$ is compact. The causality conditions as above can be arranged in the following implications: global hyperbolicity $\Rightarrow$ causal simplicity $\Rightarrow$ causal continuity $\Rightarrow$ stable causality $\Rightarrow$ strong causality $\Rightarrow$ distinguishing. The respective converse implications are all false.

An open subset P (respectively, F) of space-time M is called a *past* (respectively, *future*) *set* if $P = I^-(P)$ (respectively, $F = I^+(F)$). A nonempty past subset of a space-time is called an *indecomposable past set* if it cannot be expressed as a union of two proper past subsets. An *indecomposable future set* is defined dually. The indecomposable past sets so defined are divided into two classes, consisting of those which are the chronological pasts of a single point (these sets are called *proper indecomposable past* (*PIP*) *sets*) and those which are not (these sets are called *terminal indecomposable past* (*TIP*) *sets*). *Proper indecomposable future* (*PIF*) *sets* and *terminal indecomposable future* (*TIF*) *sets* are defined dually.

A *causal space* is a set X equipped with two relations, $\ll$ and $\leqslant$, between the points of X, such that (1) $\leqslant$ is a reflexive partial ordering (2) $\ll$ is anti-reflexive (3) $x \ll y$ implies $x \leqslant y$ and (4) either $x \ll y \leqslant z$ or $x \leqslant y \ll z$ implies $x \ll y$. Any space-time is a causal space with respect to its causal relations. The *Alexandrov topology* in a causal space X is the coarest topology in which $I^+(x)$ and $I^-(x)$ are open for all $x \in M$, where $I^+(x) = \{y \in X | x \ll y\}$ and $I^-(x) = \{y \in X | y \ll x\}$.

Dual results will often be taken for granted. All of the other terminologies and concepts will be referred to Beem, Ehrlich and Easley [1].

2. UTI space-time

A space-time M is *UTI* if it has a unique TIP set and a unique TIF set.

The following lemma is useful to represent TIP sets. This result is due to Theorem 2.1 and 2.3 of Geroch, Kronheimer and Penrose [3].

Lemma 2.1. *A subset P of a strongly causal space-time M is a terminal indecomposable past set if and only if there exists a future-directed and future-inextendible timelike* (*nonspacelike*) *curve γ in M such that $P = I^-(\gamma)$.*

From Lemma 2.1, we can have the following immediately.

Lemma 2.2. *Let M be a strongly causal space-time. Then M has a unique TIP set if and only if M itself is the unique TIP set.*

A curve γ is a *limit curve* of the sequence $\{\gamma_n\}$ in a space-time M if there is a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ such that for all p in the image of γ, each neighborhood of p intersects all but a finite number of curves of the sequence $\{\gamma_{n_k}\}$.

We note that if γ is a limit curve of the sequence$\{\gamma_n\}$ of nonspacelike curves in a strongly causal space-time M, then γ is also nonspacelike and we state an important result which is given in [1] and useful to treat Theorem 2.5.

Lemma 2.3. *Let $\{\gamma_n\}$ be a sequence of future inextendible nonspacelike curves in a space-time M. If p is an accumulation point of the sequence $\{\gamma_n\}$, then there is a nonspacelike limit curve γ of the sequence $\{\gamma_n\}$ such that $p \in \gamma$ and γ is future inextendible.*

We now consider the concept of convergence in the C^0 topology. Let γ and all curves of the sequence $\{\gamma_n\}$ be defined on the closed interval $[a, b]$. The sequence $\{\gamma_n\}$ is said to *converge to γ in the C^0 topology on curves* if $\gamma_n(a) \to \gamma(a)$, $\gamma_n(b) \to \gamma(b)$, and given any open set V containing γ, there is an integer n_0 such that $\gamma_n \subset V$ for all $n \geq n_0$. In a strongly causal space-time, these two types of convergences for sequences of nonspacelike curves are almost equivalent, which is precisely stated as following [1].

Lemma 2.4. *Let M be a strongly causal space-time. Suppose that $\{\gamma_n\}$ is a sequence of nonspacelike curves defined on $[a, b]$ such that $\gamma_n(a) \to p$ and $\gamma_n(b) \to q$. A nonspacelike curve $\gamma : [a, b] \to M$ with of $\gamma(a) = p$ and $\gamma(b) = q$ is a limit curve of $\{\gamma_n\}$ if and only if there is a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ which converges to γ in the C^0 topology on curves.*

Using the above lemmas, we are going to show the following result.

Theorem 2.5. *If a strongly causal space-time M has a unique TIP set or a unique TIF set, then M is globally hyperbolic.*

Proof. Suppose that M has a unique TIP set and p, q are any points of M such that $p < q$. Let $\{x_n\}$ be an infinite sequence in $J^+(p) \cap J^-(q)$ and let $\{\gamma_n\}$ be a sequence of nonspacelike curves from p to q such that γ_n passes through x_n. Then $\{\gamma_n - q\}$ is a sequence of future inextendible nonspacelike curves in the space-time $M - \{q\}$ as an open set of M. By Lemma 2.3, there is a nonspacelike limit curve

γ of the sequence $\{\gamma_n - q\}$ such that γ starts at p and is future inextendible in $M - \{q\}$. The curve γ can be extended to q which is a future endpoint in M. If not, γ is future inextendible nonspacelike curve in M such that $\gamma \subset J^-(q)$. $M = I^-(\gamma)$ by Lemma 2.2, and so $M \subset I^-(q)$. It contradicts to the strong causality of M. Hence by Lemma 2.4, there is a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ which converges to the curve γ, which is also considered as the extended curve to q, in the C^0 topology on curves. Let U be a neighborhood of γ in M such that $\bar{U}$ is compact. Then U contains all γ_{n_k} and so U contains all x_{n_k} for k sufficiently large. Hence there is a point $x \in U$ which is a limit point of the subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Clearly $x \in \gamma$ and $x \in J^+(p) \cap J^-(q)$. Thus $J^+(p) \cap J^-(q)$ is compact.

3. Warped product as UTI space-time

Throughout this section we shall consider Loreantzian warped products of the form $(a, b) \times_f F$ where (F, g_F) is a Riemannian manifold and (a, b), $-\infty \leq a < b \leq \infty$, has the negative definite metric $-dt^2$. Hence the metric tensor g of $(a, b) \times_f F$ is given by $g = -dt^2 \oplus f g_F$. Regard this warped product as a space-time by the timelike vector field on M metrically equivalent to dt. In particular, a Robertson-Walker space-time is a special space-time which can be written in the form $(a, b) \times_f F$ [1].

We introduce two lemmas which are shown in [1], and we shall give several results which are related the UTI condition of the warped product space-time of the form $(a, b) \times_f F$.

Lemma 3.1. *Let F be a complete Riemannian manifold. If $\mu : [c, d) \to F$ is a curve with finite arclength, then there is $m \in F$ such that $\mu(t) \to m$ as $t \to d^-$.*

Lemma 3.2. *Let (F, g_F) be a Riemannian manifold and $B = (a, b)$, $-\infty \leq a < b \leq \infty$, have the negative definite metric $-dt^2$ and let $g = -dt^2 \oplus f g_F$, Then*

(1) *$(B \times_f F, g)$ is stably causal (The first projection π serves as a global time function).*
(2) *$(B \times_f F, g)$ is globally hyperbolic if and only if (F, g_F) is complete.*

Theorem 3.3. *Let (F, g_F) be a Riemannian manifold and $B = (a, b)$, $-\infty \leq a < b \leq \infty$, have the negative definite metric $-dt^2$ and let $g = -dt^2 \oplus f g_F$. If*

$M = (B \times_f F, g)$ has a unique TIP set (respectively, TIF set), then F is complete and $b = \infty$ (respectively, $a = -\infty$).

Proof. If $M = (B \times_f F, g)$ has a unique TIP set, by Theorem 2.5, $(B \times_f F, g)$ is globally hyperbolic, so from Lemma 3.2, F is a complete Riemannian manifold. Suppose that $b < \infty$ and let p_1 and p_2 be distinct points of F. If we consider $\gamma_1 : [a_1, b) \to M$ and $\gamma_2 : [a_2, b) \to M$ given by $\gamma_1(t) = (t, p_1)$, and $\gamma_2(t) = (t, p_2)$ respectively. Then γ_1 and γ_2 are future-inextedible timelike curves in M. Moreover, $I^-(\gamma_1)$ and $I^-(\gamma_2)$ are different TIP sets in M. It contradicts. Thus $b = \infty$.

Theorem 3.4. *Let (F, g_F) be a complete Riemannian manifold with finite diameter and (a, b), $-\infty \leq a < b \leq \infty$, have the negative definite metric $-dt^2$ and let $f : (a, b) \to (0, \infty)$ be bounded by positive numbers.*

(1) *If $b = \infty$, then $M = (a, b) \times_f F$ has a unique TIP set.*
(2) *If $a = -\infty$, then $M = (a, b) \times_f F$ has a unique TIF set.*

Proof. Suppose that f is bounded by positive numbers r_1 and r_2 ($r_1 < r_2$), and let δ denote the diameter of F with respect to the Riemannian distance function d_0 induced by g_F.

(1) Let P be any TIP set in M and let $\mu : [a_0, b_0) \to M$ be a future-directed and future-inextendible timelike curve such that $P = I^-(\mu)$. It suffices to show that $P = M$ by Lemma 2.2. Reparametrizing μ by $v = (\pi \circ \mu)^{-1}(u)$, we have a future-directed and future-inextendible timelike curve γ defined on $[a_1, b_1) \subseteq (a, \infty)$ with $\pi(\gamma(u)) = u$. We claim that $b_1 = \infty$. Since

$$g(\gamma\prime(u), \gamma\prime(u)) = -1 + f(u) g_F(d\sigma(\gamma\prime(u)), d\sigma(\gamma\prime(u))) < 0,$$

$$g_F(d\sigma(\gamma\prime(u), d\sigma(\gamma\prime(u))) \leq 1/r_1.$$

If $b_1 < \infty$, the arc-length of $\sigma \circ \gamma : [a_1, b_1) \to F$ in F is equal to or less than $(b_1 - a_1)/\sqrt{r_1}$ and so the arc-length of $\sigma \circ \gamma$ is finite. By lemma 3.1. there exists $p_0 \in F$ such that $(\sigma \circ \gamma)(u) \to p_0$ as $u \to b_1^-$. Therefore $(b_1, p_0) \in M$ is the future endpoint of γ. It contradicts. Hence $b_1 = \infty$. For arbitrary point $p = (u_0, m) \in M$, let $q = \gamma(\pi(p) + (r_2 + 1)\delta)$ and choose the minimizing geodesic segment α in F with unit speed from $\alpha(0) = \sigma(p)$ to $\sigma(q)$. Now define $\beta : [0, (r_2 + 1)\delta] \to M$ by

$$\beta(u) = (u_0 + u, \alpha(\frac{d_0(\sigma(p), \sigma(q))}{(r_2 + 1)\delta} u)).$$

Then $\beta(0) = (\pi(p), \sigma(p)) = p$,

$$\begin{aligned}\beta((r_2+1)\delta) &= (\pi(p) + (r_2+1)\delta, \sigma(\gamma(\pi(p) + (r_2+1)\delta))) \\ &= \gamma(\pi(p) + (r_2+1)\delta) \\ &\in \gamma([a_1, \infty)).\end{aligned}$$

On the other hand,

$$g(\beta'(u), \beta'(u)) = -1 + f(u_0+u)[\frac{d_0(\sigma(p), \sigma(q))}{(r_2+1)\delta}]^2 < 0,$$

$$g(\beta'(u), \gamma'(u_0+u)) \leq -1 + f(u_0+u) g_F(d\sigma(\beta'(u)), d\sigma(\gamma'(u_0+u)))$$

$$\begin{aligned}&\leq -1 + f(u_0+u)[g_F(d\sigma(\gamma'(u_0+u)), d\sigma(\gamma'(u_0+u)))]^{\frac{1}{2}}[\frac{d_0(\sigma(p), \sigma(q))}{(r_2+1)\delta}] \\ &\leq -1 + \frac{f(u_0+u)}{r_2+1}[g_F(d\sigma(\gamma'(u_0+u)), d\sigma(\gamma'(u_0+u)))]^{\frac{1}{2}} \\ &< 0.\end{aligned}$$

Hence β is a future-directed timelike curve in M, and

$$p = \beta(0) \in I^-(\beta((r_2+1)\delta)) \subset I^-(\gamma).$$

Therefore $P = I^-(\mu) = I^-(\gamma) = M$. It completes (1).

(2) It is obtained by the similar method to (1).

Corollary 3.5. *Let (F, g_F) be a compact Riemannian manifold and let f be a smooth function on $\mathbb{R}$ bounded by positive numbers, let $M = (\mathbb{R} \times_f F, -dt^2 \oplus f g_F)$. Then M is UTI.*

The Einstein static universe is a simplest example of a Robertson-Walker space-time and the n-dimensional Einstein static universe is defined as the warped product space-time $\mathbb{R} \times_f F$, where $\mathbb{R}$ has negative definite metric $-dt^2$, $F = S^{n-1}$ has the standard spherical Riemannian metric, and $f : \mathbb{R} \to (0, \infty)$ is the trivial warping function $f = 1$. Thus we have the following corollary.

Corollary 3.6. *The n-dimensional Einstein static universe is an UTI space-time.*

4. Causal boundary of UTI space-time

In this section, any space-time M is assumed to be strongly causal.

To introduce some kind of boundary for any space-time (M, g) one has to an enlarged topological space $\bar{M}$ and an open dense embedding i and the set $\bar{M} - i(M)$ is interpreted as the boundary of M. Various constructions have been put forward. The causal boundary construction given by Geroch, Kronheimer and Penrose [3] makes use only of the causal structure of the space-time and thus it is conformally invariant. However it is difficult to construct Hausdorff topological space $\bar{M}$ as a causal space. The causal boundary by Geroch et al is obtained from constructing the space $\bar{M}$ which consists of the future and past endpoints of all nonspacelike curves and is a Hausdorff space. Here M is open dense embedded in this space $\bar{M}$.

Denote by $\hat{M}$ the collection of all indecomposable past sets of a space-time M, and $\check{M}$ the collection of all indecomposable future sets of M. Let M^* be the space by taking the union $\hat{M} \cup \check{M}$ and identifying $I^+(p) \in \hat{M}$ with $I^-(p) \in \check{M}$ for each $p \in M$. The element of M^* corresponding to an element Q of $\hat{M} \cup \check{M}$ is written as Q^*. The topology τ^* on M^* consists of the sets defined to be the unions and finite intersections subsets of the form P^{int}, P^{ext}, F^{int} and F^{ext}, where $P \in \hat{M}$, $F \in \check{M}$, and

$$P^{int} = \{Q^* | Q \in \check{M} \text{ and } P \cap Q \neq \emptyset\}$$

$$P^{ext} = \{Q^* | Q \in \check{M} \text{ and if } Q = I^+(S) \text{ for some } S \subseteq M,$$

$$\text{then } I^-(S) \not\subseteq P\}.$$

F^{int} and F^{ext} are defined dually, with the roles of past and future interchanged. The pair $(\bar{M}, \bar{\tau})$ is the quotient space obtained from the space (M^*, τ^*) by identifying the smallest number of points of M^* necessary to obtain a Hausdorff space. $\bar{M}$ contains the causal boundary $\partial_c M$ consisting of TIPs and TIFs where identifications has been made as above.

In this section, for UTI space-times we consider the causal boundary constructed by Geroch, Kronheimer and Penrose [3].

Throughout the rest of this section, we assume that M is UTI. Let $q^* = \{I^+(q), I^-(q)\}$ and let $\hat{\infty}$ denote the unique TIP set and $\check{\infty}$ denote the unique TIF set of M. The following lemma can be shown easily from the definitions of int-, ext-sets as given above.

$$I^+(p)^{int} = \{q^*, \hat{\infty} | p \ll q\}, \; I^-(p)^{int} = \{q^*, \check{\infty} | q \ll p\},$$

$$I^+(p)^{ext} = \{q^*, \check{\infty} | q \not\leqslant p\}, \; I^-(p)^{ext} = \{q^*, \hat{\infty} | p \not\leqslant q\}$$

$$\hat{\infty}^{int} = M^* - \{\hat{\infty}\}, \; \check{\infty}^{int} = M^* - \{\check{\infty}\}, \; \hat{\infty}^{ext} = \emptyset, \; \check{\infty}^{ext} = \emptyset.$$

We define relations $\ll$ and $\leqslant$ in M^* as follows:

$$P^* \ll Q^* \text{ iff } Q^* \in P^{int} \text{ or } P^* \in Q^{int}$$

$$P^* \leqslant Q^* \text{ iff } Q^* \notin P^{ext} \text{ or } P^* \notin Q^{ext}$$

Lemma 4.1. *For arbitrary points p, q in a strongly causal UTI space-time M,*

(1) *$p^* \ll q^*$ iff $p \ll q$ in M,*
(2) *$\check{\infty} \ll p^* \ll \hat{\infty}$,*
(3) *$p^* \leqslant q^*$ iff $p \leqslant q$ in M.*

For $P^* \in M^*$, define

$$I^+(P^*) = \{Q^* \in M^* | P^* \ll Q^*\}, \; I^-(P^*) = \{Q^* \in M^* | Q^* \ll P^*\}$$

$$J^+(P^*) = \{Q^* \in M^* | P^* \leqslant Q^*\}, \; J^-(P^*) = \{Q^* \in M^* | Q^* \leqslant P^*\}$$

From Lemma 4.1, we obtain the following.

Lemma 4.2. *Let M be a strongly causal UTI space-time. Then*

(1) *For $p \in M$, $\{I^+(a^*) \cap I^-(b^*) | a \ll p \ll b\}$ forms a neighborhood base of p^* in (M^*, τ^*).*
(2) *$\{I^+(p^*) | p \in M\}$ forms a neighborhood base of $\hat{\infty}$ in (M^*, τ^*).*
(3) *$\{I^-(p^*) | p \in M\}$ forms a neighborhood base of $\check{\infty}$ in (M^*, τ^*).*

These lemmas give rise to the following results.

Theorem 4.3. *Let M be a strongly causal UTI space-time. Then*

(1) *M^* is a causal space relative to the above relations $\ll$ and $\leqslant$.*
(2) *(M^*, τ^*) is a Hausdorff space and τ^* is the Alexandrov topology of M^* as a causal space.*

From Theorem 4.3, further identification is not required for Hausdorff condition. Thus we regard the space $\bar{M}$ as the same topological space as M^* and the same causal space as M^*. Since M satisfies strong causality, the space-time topology is equal to the Alexandrov topology induced by the causal relations of M [8]. τ^* is considered as the extended Alexandrov topology to the space attached boundary.

Theorem 4.4. *Let M be a strongly causal UTI space-time. Then*

(1) $i : (M, \tau) \longrightarrow (\bar{M}, \bar{\tau})$ *given by* $i(p) = p^*$ *for* $p \in M$, *is an open dense embedding and so* $\bar{M} = i(M) \cup \partial_c M$, *where* $\partial_c M = \{\check{\infty}, \hat{\infty}\}$.

(2) *If* γ *is any inextendible nonspacelike curve in* M, *then* $i \circ \gamma$ *is a curve in* $(\bar{M}, \bar{\tau})$ *with* $\check{\infty}$ *initial endpoint and* $\hat{\infty}$ *the terminal endpoint.*

Corollary 4.5. *Let* (F, g_F) *be a compact Riemannian manifold and let* f *be a smooth function on* $\mathbb{R}$ *bounded by positive numbers, let* $M = (\mathbb{R} \times_f F, -dt^2 \oplus f g_F)$. *Then* $\bar{M}$ *is a compact space.*

Proof. Let $\bar{\mathbb{R}} = \{\infty, -\infty\} \cup \mathbb{R}$ be the extended real space. Since $\bar{\mathbb{R}}$ is homeomorphic to the closed interval $[0, 1]$, th product space $\bar{\mathbb{R}} \times F$ is compact. Define an equivalence relation in $\bar{\mathbb{R}} \times F$ by $(s, a) \sim (t, b)$ iff $[s = t$ and $a = b]$ or $[s = t = \infty]$ or $[s = t = -\infty]$. Then the quotient space by this equivalence relation is compact and it is homeomorphic to $\bar{M}$.

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