J. Korea Soc. Math. Ed. Ser. B: Pure Appl. Math.
5(1998), no. 1, 79–86

AN INTEGRAL FORMULA ON CONTACT CR-SUBMANIFOLDS OF AN ODD-DIMENSIONAL UNIT SPHERE

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ABSTRACT. In this paper, we study an $(n+1)$-dimensional compact, orientable, minimal contact CR-submanifold of $(n-1)$ ontact CR-dimension in a $(2m+1)$-dimensional unit sphere S^{2m+1} in terms of integral formula.

1. Introduction

Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere, that is,

$$S^{2m+1} = \{z \in \mathbb{C}^{m+1} \ : \ \|z\| = 1\}.$$

For any point $z \in S^{2m+1}$ we put $\xi = Jz$, where J denotes the almost complex structure of $\mathbb{C}^{m+1}$. We consider the orthogonal projection $\pi : T_z\mathbb{C}^{m+1} \to T_zS^{2m+1}$. Putting $\phi = \pi \circ J$, we can see that the aggregate (ϕ, ξ, η, g) is a Sasakian structure on S^{2m+1}, where η is a 1-form dual to ξ and g the standard metric tensor field on S^{2m+1}. So S^{2m+1} can be considered as a Sasakian manifold of constant ϕ-sectional curvature 1, that is, of constant curvature 1 (cf. [11]).

Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of S^{2m+1} and denote by $\mathcal{D}_x$ the ϕ-invariant subspace $\phi T_xM \cap T_xM$ of the tangent space T_xM of M at x in M. Then ξ cannot be contained in $\mathcal{D}_x$ at any point x in M (see section 2). If there is a non-zero vector U which is orthogonal to ξ and contained in the complementary orthogonal subspace $\mathcal{D}_x^\perp$ to $\mathcal{D}_x$ in T_xM, then ϕU must be normal to M. Thus the assumption $dim\mathcal{D}_x^\perp$ being constant and greater than or equal to 2 at each point x in M yields that M can be dealt with a contact CR-submanifold in the sense of Yano-Kon [11].

Received by the editors January 22, 1998 and, in revised form April 30, 1998.
1991 *Mathematics Subject Classifications.* 53C40 , 53C42.
Key words and phrases. Contact CR-submanifpld, Odd-dimensional unit sphere, Sasakian manifold, Minimal.

In the present paper we assume that $dim\mathcal{D}_x = n-1$, that is, $dim\mathcal{D}_x^{\perp} = 2$ at each point x in M and determine such submanifolds. Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class C^∞.

Recently, Kwon and Pak [8] proved the following :

Theorem A. *Let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in a $(2m+1)$-dimensional unit sphere S^{2m+1}. If $A_1F = FA_1$ and N_1 is parallel with respect to the normal connection, then M is locally a product $M_1 \times M_2$, where M_1 and M_2 belong to some odd-dimensional spheres, $N_1 := \phi U$ and A_1 is the shape operator corresponding to N_1.*

The purpose of the present paper is to give another characterization of an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a $(2m+1)$-dimensional unit sphere S^{2m+1} by using the following integral formula due to Yano [10] :

$$\int_M \{Ric(X,X) + \frac{1}{2}\|\mathcal{L}_X g\|^2 - \|\nabla X\|^2 - (divX)^2\} * 1 = 0, \tag{1.1}$$

where X is an arbitrary tangent vector field on M, $\mathcal{L}_X$ the Lie derivative with respect to X, ∇ the Riemannian connection induced on M, $*1$ the volume element of M and $\|Y\|$ the length with respect to the Riemannian metric of a vector field Y on M.

2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure (ϕ, ξ, η, g). Then by definition it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to $\overline{M}$. We consider a Riemannian manifold isometrically immersed in $\overline{M}$ with induced metric tensor field g.

First of all we note that an n-dimensional submanifold normal to the structure vector field ξ of $\overline{M}$ is anti-invariant with respect to ϕ, that is, $\phi T_xM \subset T_xM^\perp$ for each point x of M, and $m \geq n$ (for details, see Yano-Kon [11]).

In the sequel we assume that M is an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of a $(2m+1)$-dimensional almost contact metric manifold $\overline{M}$. We now denote by $\mathcal{D}_x$ the ϕ-invariant subspace defined by $T_xM \cap \phi T_xM$ and by $\mathcal{D}_x^\perp$ the complementary orthogonal to $\mathcal{D}_x$ in T_xM. Then the sturcture vector field ξ is contained in $\mathcal{D}_x^\perp$. In fact, if $\xi \in \mathcal{D}_x$, then there is a vector field X tangent to M such that $\xi = \phi X$, from which applying the operator ϕ and using (2.1), we have $X = \eta(X)\xi$. Thus it follows that $\xi = 0$, which is a contradiction. Hence $\xi \in \mathcal{D}_x^\perp$ at each point x of M. Moreover by definition we can easily see that $\phi\mathcal{D}_x^\perp \subset T_xM^\perp$ for each point x of M.

If the ϕ-invariant subspace $\mathcal{D}_x$ has constant dimension for x in M, then M is called a *contact CR-submanifold* (cf. [1], [11]) and the constant is called *contact CR-dimension* of M (see Kwon and Pak [8]).

3. Fundamental properties of contact CR-submanifolds

Let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in a $(2m+1)$-dimensional almost contact metric manifold $\overline{M}$. Then by definition $dim\mathcal{D}_x^\perp = 2$ for each point x in M, and so there is a unit vector field U contained in $\mathcal{D}^\perp$ which is orthogonal to ξ. Since $\phi\mathcal{D}^\perp \subset TM^\perp$, ϕU is a unit normal vector field to M, which will be denoted by N_1, that is,

$$N_1 = \phi U. \tag{3.1}$$

Moreover it is clear that $\phi TM \subset TM \oplus Span\{N_1\}$. Hence we have, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha,\ \alpha = 1, \dots, p\}$ $(p = 2m - n)$ of normal vectors to M, the following decomposition in tangential and normal components :

$$\phi X = FX + u^1(X)N_1, \tag{3.2}$$

$$\phi N_\alpha = -U_\alpha + PN_\alpha, \quad \alpha = 1, \dots, p. \tag{3.3}$$

It is easily shown that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Since the structure vector field ξ is tangent to M, (2.1) implies

$$g(FU_\alpha, X) = -u^1(X)g(N_1, PN_\alpha), \tag{3.4}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(PN_\alpha, PN_\beta). \tag{3.5}$$

We also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha} \tag{3.6}$$

and consequently

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \ldots, p. \tag{3.7}$$

Furthermore from (3.2) it is clear that

$$F\xi = 0, \quad u^1(\xi) = 0, \quad FU = 0, \quad u^1(U) = 1. \tag{3.8}$$

Next, applying ϕ to (3.1) and using (2.1) and (3.3), we have

$$U_1 = U, \quad PN_1 = 0. \tag{3.9}$$

Applying ϕ to (3.2) and using (2.1), (3.2), (3.3) and (3.9), we also have

$$F^2X = -X + \eta(X)\xi + u^1(X)U, \quad u^1(FX) = 0. \tag{3.10}$$

On the other hand, it follows from (3.3), (3.7) and (3.9) that

$$\phi N_1 = -U, \quad \phi N_\alpha = PN_\alpha, \quad \alpha = 2, \ldots, p. \tag{3.11}$$

and moreover we may put

$$PN_\alpha = \sum_{\beta=2}^{p} P_{\alpha\beta} N_\beta, \quad \alpha = 2, \ldots, p, \tag{3.12}$$

where $(P_{\alpha\beta})$ is a skew-symmetric matrix which satisfies

$$\sum_{\beta=2}^{p} P_{\alpha\beta} P_{\beta\gamma} = -\delta_{\alpha\gamma}. \tag{3.13}$$

We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\overline{M}$ and M, respectively and denote by D the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. The Gauss and Weingartan equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3.14}$$

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + D_X N_\alpha, \quad \alpha = 1, \ldots, p \tag{3.15}$$

for any vector fields X, Y tangent to M. Here h denotes the second fundamental form and A_α is the shape operator corresponding to N_α. They are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y) N_\alpha.$$

Furthermore we put

$$D_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X) N_\beta, \tag{3.16}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of D. Finally, the equation of Gauss, Codazzi and Ricci are respectively given as follows (cf. [2,5,8]) :

$$\begin{aligned}\overline{R}(X,Y)Z =& R(X,Y)Z \\ &+ \sum_\alpha \{g(A_\alpha X, Z)A_\alpha Y - g(A_\alpha Y, Z)A_\alpha X\},\end{aligned} \tag{3.17}$$

$$\begin{aligned}g(\overline{R}(X,Y)Z, N_\alpha) =& g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z) \\ &+ \sum_\beta \{g(A_\beta Y, Z)s_{\beta\alpha}(X) - g(A_\beta X, Z)s_{\beta\alpha}(Y)\},\end{aligned} \tag{3.18}$$

$$g(\overline{R}(X,Y)N_\alpha, N_\beta) = g(R^\perp(X,Y)N_\alpha, N_\beta) + g([A_\beta, A_\alpha]X, Y) \tag{3.19}$$

for any vector fields X, Y, Z tangent to M, where $\overline{R}$ and R denote the Riemannian curvature tensor of $\overline{M}$ and M, respectively and $R^\perp$ is the curvature tensor of the normal connection D.

4. Main results

In this section we specialize to the case of an ambient Sasakian manifold $\overline{M}$, that is,

$$\overline{\nabla}_X \xi = \phi X, \tag{4.1}$$

$$(\overline{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{4.2}$$

Then, by differentiating (3.2) and (3.11) covariantly and by comparing the tangential and normal parts, we have

$$(\nabla_Y F)X = -g(Y,X)\xi + \eta(X)Y - g(A_1 Y, X)U + u^1(X)A_1 Y, \tag{4.3}$$

$$(\nabla_Y u^1)X = g(FA_1 Y, X), \tag{4.4}$$

$$\nabla_X U = FA_1 X, \tag{4.5}$$

$$g(A_\alpha U, X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P_{\beta\alpha}, \quad \alpha = 2, \ldots, p(= 2m-n). \tag{4.6}$$

On the other hand, since ξ is tangent to M, (4.1) gives

$$\nabla_X \xi = FX, \tag{4.7}$$

$$g(A_1\xi, X) = u^1(X), \quad \text{that is,} \quad A_1\xi = U, \tag{4.8}$$

$$A_\alpha \xi = 0, \quad \alpha = 2, \ldots, p. \tag{4.9}$$

We suppose that $\overline{M}$ is a Sasakian manifold of constant ϕ-sectional curvature 1, that is, of constant curvature 1 and that N_1 is parallel with respect to the normal connection D. Hence it follows from (3.16) that

$$s_{1\beta} = 0, \quad \beta = 2, \ldots, p, \tag{4.10}$$

which and (4.6) give

$$A_\alpha U = 0, \quad \alpha = 2, \ldots, p. \tag{4.11}$$

Next, since the curvature tensor $\overline{R}$ has the form

$$\overline{R}(\overline{X},\overline{Y})\overline{Z} = g(\overline{Y},\overline{Z})\overline{X} - g(\overline{X},\overline{Z})\overline{Y}$$

for $\overline{X}$, $\overline{Y}$, $\overline{Z}$ tangent to $\overline{M}$, the equations (3.17), (3.18) and (3.19) imply

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y \\ &+ \sum_\alpha \{g(A_\alpha Y, Z)A_\alpha X - g(A_\alpha X, Z)A_\alpha Y\}, \end{aligned} \tag{4.12}$$

$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = 0, \tag{4.13}$$

$$g(R^\perp(X,Y)N_\alpha, N_\beta) = g([A_\beta, A_\alpha]X, Y), \tag{4.14}$$

$$Ric(X,Y) = ng(X,Y) + \sum_\alpha \{(trA_\alpha)g(A_\alpha X, Y) - g(A_\alpha^2 X, Y)\} \tag{4.15}$$

with the help of (4.10).

As an application of the integral formula (1.1), we will prove

Theorem 1. *Let M be an $(n+1)$-dimensional compact, orientable, minimal contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1}. If the normal vector field N_1 is parallel with respect to the normal connection, then*

$$\int_M trA_1^2 * 1 \geqq (n+1)Vol(M). \tag{4.16}$$

Proof. Putting $X = U$ in (1.1) gives

$$\int_M \{Ric(U,U) + \frac{1}{2}\|\mathcal{L}_U g\|^2 - \|\nabla U\|^2 - (divU)^2\} * 1 = 0. \tag{4.17}$$

On the other hand, the Ricci equation (4.15) together with (4.11) yields

$$Ric(U,U) = n + (trA_1)g(A_1U,U) - g(A_1^2U,U). \tag{4.18}$$

From (4.5) it follows that

$$divU = tr(FA_1) = 0. \tag{4.19}$$

We have from (4.5)

$$(\mathcal{L}_U g)(X,Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X) = g((FA_1 - A_1F)X, Y). \tag{4.20}$$

And using (3.9), (3.10), (4.5) and (4.8), we get

$$\|\nabla U\|^2 = trA_1^2 - 1 - g(A_1^2U,U). \tag{4.21}$$

Since M is minimal, $trA_\alpha = 0$, $\alpha = 1, \ldots, p$. Therefore substituting (4.18), (4.19) and (4.21) into (4.17), we obtain

$$\int_M \{\frac{1}{2}\|\mathcal{L}_U g\|^2 + (n+1) - trA_1^2\} * 1 = 0. \tag{4.22}$$

Thus we have the inequality (4.16). □

From Theorem A, we have immediately

Corollary 2. *Let M be as in Theorem 1. If the normal vector field N_1 is parallel with respect to the normal connection and*

$$\int_M trA_1^2 * 1 = (n+1)Vol(M),$$

then M is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres.

Proof. From the hypotheses and (4.22), we have $\|\mathcal{L}_U g\|^2 = 0$ and consequently $A_1F = FA_1$ because of (4.20). Combining Theorem A and $A_1F = FA_1$, we have the required result in Corollary 2. □

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