AN INTEGRAL FORMULA ON CONTACT CR-SUBMANIFOLDS OF AN ODD-DIMENSIONAL UNIT SPHERE

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ABSTRACT. In this paper, we study an \((n + 1)\)-dimensional compact, orientable, minimal contact CR-submanifold of \((n - 1)\) contact CR-dimension in a \((2m + 1)\)-dimensional unit sphere \(S^{2m+1}\) in terms of integral formula.

1. Introduction

Let \(S^{2m+1}\) be a \((2m + 1)\)-dimensional unit sphere, that is,
\[
S^{2m+1} = \{ z \in \mathbb{C}^{m+1} : \|z\| = 1 \}.
\]

For any point \(z \in S^{2m+1}\) we put \(\xi = Jz\), where \(J\) denotes the almost complex structure of \(\mathbb{C}^{m+1}\). We consider the orthogonal projection \(\pi : T_z \mathbb{C}^{m+1} \to T_z S^{2m+1}\).

Putting \(\phi = \pi \circ J\), we can see that the aggregate \((\phi, \xi, \eta, g)\) is a Sasakian structure on \(S^{2m+1}\), where \(\eta\) is a 1-form dual to \(\xi\) and \(g\) the standard metric tensor field on \(S^{2m+1}\). So \(S^{2m+1}\) can be considered as a Sasakian manifold of constant \(\phi\)-sectional curvature 1, that is, of constant curvature 1 (cf. [11]).

Let \(M\) be an \((n + 1)\)-dimensional submanifold tangent to the structure vector field \(\xi\) of \(S^{2m+1}\) and denote by \(D_x\) the \(\phi\)-invariant subspace \(\phi T_x M \cap T_x M\) of the tangent space \(T_x M\) of \(M\) at \(x\) in \(M\). Then \(\xi\) cannot be contained in \(D_x\) at any point \(x\) in \(M\) (see section 2). If there is a non-zero vector \(U\) which is orthogonal to \(\xi\) and contained in the complementary orthogonal subspace \(D_x^\perp\) to \(D_x\) in \(T_x M\), then \(\phi U\) must be normal to \(M\). Thus the assumption \(dim D_x^\perp\) being constant and greater than or equal to 2 at each point \(x\) in \(M\) yields that \(M\) can be dealt with a contact CR-submanifold in the sense of Yano-Kon [11].

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In the present paper we assume that $\dim D_x = n - 1$, that is, $\dim D_x^\perp = 2$ at each point $x$ in $M$ and determine such submanifolds. Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class $C^\infty$.

Recently, Kwon and Pak [8] proved the following:

**Theorem A.** Let $M$ be an $(n + 1)$-dimensional contact CR-submanifold of $(n - 1)$ contact CR-dimension in a $(2m + 1)$-dimensional unit sphere $S^{2m+1}$. If $A_1 F = FA_1$ and $N_1$ is parallel with respect to the normal connection, then $M$ is locally a product $M_1 \times M_2$, where $M_1$ and $M_2$ belong to some odd-dimensional spheres, $N_1 := \phi U$ and $A_1$ is the shape operator corresponding to $N_1$.

The purpose of the present paper is to give another characterization of an $(n + 1)$-dimensional contact CR-submanifold of $(n - 1)$ contact CR-dimension immersed in a $(2m + 1)$-dimensional unit sphere $S^{2m+1}$ by using the following integral formula due to Yano [10]:

$$\int_M \{Ric(X, X) + \frac{1}{2} \|\mathcal{L}_X g\|^2 - \|\nabla X\|^2 - (\text{div} X)^2\} \ast 1 = 0,$$

(1.1)

where $X$ is an arbitrary tangent vector field on $M$, $\mathcal{L}_X$ the Lie derivative with respect to $X$, $\nabla$ the Riemannian connection induced on $M$, $\ast 1$ the volume element of $M$ and $\|Y\|$ the length with respect to the Riemannian metric of a vector field $Y$ on $M$.

2. Preliminaries

Let $\overline{M}$ be a $(2m + 1)$-dimensional almost contact metric manifold with structure $(\phi, \xi, \eta, g)$. Then by definition it follows that

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

(2.1)

for any vector fields $X, Y$ tangent to $\overline{M}$. We consider a Riemannian manifold isometrically immersed in $\overline{M}$ with induced metric tensor field $g$.

First of all we note that an $n$-dimensional submanifold normal to the structure vector field $\xi$ of $\overline{M}$ is anti-invariant with respect to $\phi$, that is, $\phi T_x M \subset T_x M^\perp$ for each point $x$ of $M$, and $m \geq n$ (for details, see Yano-Kon [11]).
In the sequel we assume that $M$ is an $(n+1)$-dimensional submanifold tangent to
the structure vector field $\xi$ of a $(2m+1)$-dimensional almost contact metric manifold $\overline{M}$. We now denote by $D_x$ the $\phi$-invariant subspace defined by $T_x M \cap \phi T_x M$ and by $D_x^\perp$ the complementary orthogonal to $D_x$ in $T_x M$. Then the structure vector field $\xi$ is contained in $D_x^\perp$. In fact, if $\xi \in D_x$, then there is a vector field $X$ tangent to
$M$ such that $\xi = \phi X$, from which applying the operator $\phi$ and using (2.1), we have
$X = \eta(X)\xi$. Thus it follows that $\xi = 0$, which is a contradiction. Hence $\xi \in D_x^\perp$ at
each point $x$ of $M$. Moreover by definition we can easily see that $\phi D_x^\perp \subset T_x M^\perp$ for
each point $x$ of $M$.

If the $\phi$-invariant subspace $D_x$ has constant dimension for $x$ in $M$, then $M$ is
called a contact CR-submanifold (cf. [1], [11]) and the constant is called contact
CR-dimension of $M$ (see Kwon and Pak [8]).

3. Fundamental properties of contact CR-submanifolds

Let $M$ be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact
CR-dimension in a $(2m+1)$-dimensional almost contact metric manifold $\overline{M}$. Then
by definition $\dim D_x^\perp = 2$ for each point $x$ in $M$, and so there is a unit vector field $U$
contained in $D^\perp$ which is orthogonal to $\xi$. Since $\phi D^\perp \subset TM^\perp$, $\phi U$ is a unit normal
vector field to $M$, which will be denoted by $N_1$, that is,

$$N_1 = \phi U.$$  \hspace{1cm} (3.1)

Moreover it is clear that $\phi TM \subset TM \oplus \text{Span}\{N_1\}$. Hence we have, for any tangent
vector field $X$ and for a local orthonormal basis $\{N_\alpha, \alpha = 1, \ldots, p\}$ ($p = 2m - n$) of normal vectors to $M$, the following decomposition in tangential and normal components:

$$\phi X = FX + u^1(X)N_1,$$  \hspace{1cm} (3.2)

$$\phi N_\alpha = -U_\alpha + PN_\alpha, \hspace{0.5cm} \alpha = 1, \ldots, p.$$  \hspace{1cm} (3.3)

It is easily shown that $F$ and $P$ are skew-symmetric linear endomorphisms acting
on $T_x M$ and $T_x M^\perp$, respectively. Since the structure vector field $\xi$ is tangent to
$M$, (2.1) implies

$$g(FU_\alpha, X) = -u^1(X)g(N_1, PN_\alpha),$$  \hspace{1cm} (3.4)

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(PN_\alpha, PN_\beta).$$  \hspace{1cm} (3.5)
We also have
\[ g(U_\alpha, X) = u^1(X)\delta_{1\alpha} \] (3.6)
and consequently
\[ g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \ldots, p. \] (3.7)

Furthermore from (3.2) it is clear that
\[ F^\xi = 0, \quad u^1(\xi) = 0, \quad FU = 0, \quad u^1(U) = 1. \] (3.8)

Next, applying \( \phi \) to (3.1) and using (2.1) and (3.3), we have
\[ U_1 = U, \quad PN_1 = 0. \] (3.9)

Applying \( \phi \) to (3.2) and using (2.1), (3.2), (3.3) and (3.9), we also have
\[ F^2 X = -X + \eta(X)\xi + u^1(X)U, \quad u^1(FX) = 0. \] (3.10)

On the other hand, it follows from (3.3), (3.7) and (3.9) that
\[ \phi N_1 = -U, \quad \phi N_\alpha = PN_\alpha, \quad \alpha = 2, \ldots, p. \] (3.11)

and moreover we may put
\[ PN_\alpha = \sum_{\beta = 2}^p P_{\alpha\beta}N_\beta, \quad \alpha = 2, \ldots, p, \] (3.12)

where \( (P_{\alpha\beta}) \) is a skew-symmetric matrix which satisfies
\[ \sum_{\beta = 2}^p P_{\alpha\beta}P_{\beta\gamma} = -\delta_{\alpha\gamma}. \] (3.13)

We denote by \( \nabla \) and \( \nabla \) the Levi-Civita connection on \( \overline{M} \) and \( M \), respectively and denote by \( D \) the normal connection induced from \( \nabla \) in the normal bundle \( TM^\perp \) of \( M \). The Gauss and Weingarten equations are
\[ \nabla_X Y = \nabla_X Y + h(X, Y), \] (3.14)
\[ \nabla_X N_\alpha = -A_\alpha X + D_X N_\alpha, \quad \alpha = 1, \ldots, p \] (3.15)
for any vector fields $X, Y$ tangent to $M$. Here $h$ denotes the second fundamental form and $A_\alpha$ is the shape operator corresponding to $N_\alpha$. They are related by

$$h(X, Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y) N_\alpha.$$ 

Furthermore we put

$$D_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X) N_\beta,$$  \hspace{1cm} (3.16)

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $D$. Finally, the equation of Gauss, Codazzi and Ricci are respectively given as follows (cf. [2,5,8]) :

$$\bar{R}(X, Y)Z = R(X, Y)Z$$
$$+ \sum_{\alpha} \{ g(A_\alpha X, Z) A_\alpha Y - g(A_\alpha Y, Z) A_\alpha X \},$$  \hspace{1cm} (3.17)

$$g(\bar{R}(X, Y)Z, N_\alpha) = g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z)$$
$$+ \sum_{\beta} \{ g(A_\beta Y, Z) s_{\beta\alpha}(X) - g(A_\beta X, Z) s_{\beta\alpha}(Y) \},$$  \hspace{1cm} (3.18)

$$g(\bar{R}(X, Y)N_\alpha, N_\beta) = g(R^\perp(X, Y)N_\alpha, N_\beta) + g([A_\beta, A_\alpha] X, Y)$$  \hspace{1cm} (3.19)

for any vector fields $X, Y, Z$ tangent to $M$, where $\bar{R}$ and $R$ denote the Riemannian curvature tensor of $\bar{M}$ and $M$, respectively and $R^\perp$ is the curvature tensor of the normal connection $D$.

4. Main results

In this section we specialize to the case of an ambient Sasakian manifold $\bar{M}$, that is,

$$\bar{\nabla}_X \xi = \phi X,$$  \hspace{1cm} (4.1)

$$(\bar{\nabla}_X \phi) Y = -g(X, Y) \xi + \eta(Y) X.$$  \hspace{1cm} (4.2)
Then, by differentiating (3.2) and (3.11) covariantly and by comparing the tangential and normal parts, we have

\[(\nabla_Y F)X = -g(Y, X)\xi + \eta(X)Y - g(A_1 Y, X)U + u^1(X)A_1 Y,\]  \hfill (4.3)

\[(\nabla_Y u^1)X = g(FA_1 Y, X),\]  \hfill (4.4)

\[\nabla_X U = FA_1 X,\]  \hfill (4.5)

\[g(A_\alpha U, X) = -\sum_{\beta=2}^p s_{1\beta}(X)P_{\beta\alpha}, \quad \alpha = 2, \ldots, p (= 2m - n).\]  \hfill (4.6)

On the other hand, since \(\xi\) is tangent to \(M\), (4.1) gives

\[\nabla_X \xi = FX,\]  \hfill (4.7)

\[g(A_1 \xi, X) = u^1(X), \quad \text{that is,} \quad A_1 \xi = U,\]  \hfill (4.8)

\[A_\alpha \xi = 0, \quad \alpha = 2, \ldots, p.\]  \hfill (4.9)

We suppose that \(\overline{M}\) is a Sasakian manifold of constant \(\phi\)-sectional curvature 1, that is, of constant curvature 1 and that \(N_1\) is parallel with respect to the normal connection \(D\). Hence it follows from (3.16) that

\[s_{1\beta} = 0, \quad \beta = 2, \ldots, p,\]  \hfill (4.10)

which and (4.6) give

\[A_\alpha U = 0, \quad \alpha = 2, \ldots, p.\]  \hfill (4.11)

Next, since the curvature tensor \(\overline{R}\) has the form

\[\overline{R}(\overline{X}, \overline{Y})\overline{Z} = g(\overline{Y}, \overline{Z})\overline{X} - g(\overline{X}, \overline{Z})\overline{Y}\]

for \(\overline{X}, \overline{Y}, \overline{Z}\) tangent to \(\overline{M}\), the equations (3.17), (3.18) and (3.19) imply

\[R(X, Y)Z = g(Y, Z)X - g(X, Z)Y\]

\[+ \sum_\alpha \{g(A_\alpha Y, Z)A_\alpha X - g(A_\alpha X, Z)A_\alpha Y\},\]  \hfill (4.12)

\[(\nabla_X A_1)Y - (\nabla_Y A_1)X = 0,\]  \hfill (4.13)

\[g(R^\perp(X, Y)N_\alpha, N_\beta) = g([A_\beta, A_\alpha]X, Y),\]  \hfill (4.14)

\[Ric(X, Y) = ng(X, Y) + \sum_\alpha \{(trA_\alpha)g(A_\alpha X, Y) - g(A^2_\alpha X, Y)\}\]  \hfill (4.15)

with the help of (4.10).

As an application of the integral formula (1.1), we will prove
Theorem 1. Let $M$ be an $(n+1)$-dimensional compact, orientable, minimal contact CR-submanifold of $(n - 1)$ contact CR-dimension in $S^{2m+1}$. If the normal vector field $N_1$ is parallel with respect to the normal connection, then
\[ \int_M \text{tr} A_1^2 \ast 1 \geq (n + 1) \text{Vol}(M). \] (4.16)

Proof. Putting $X = U$ in (1.1) gives
\[ \int_M \{ \text{Ric}(U, U) + \frac{1}{2} \| \mathcal{L}_U g \|^2 - \| \nabla U \|^2 - (\text{div} U)^2 \} \ast 1 = 0. \] (4.17)

On the other hand, the Ricci equation (4.15) together with (4.11) yields
\[ \text{Ric}(U, U) = n + (\text{tr} A_1) g(A_1 U, U) - g(A_1^2 U, U). \] (4.18)

From (4.5) it follows that
\[ \text{div} U = \text{tr}(F A_1) = 0. \] (4.19)

We have from (4.5)
\[ (\mathcal{L}_U g)(X, Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X) = g((F A_1 - A_1 F) X, Y). \] (4.20)

And using (3.9), (3.10), (4.5) and (4.8), we get
\[ \| \nabla U \|^2 = \text{tr} A_1^2 - 1 - g(A_1^2 U, U). \] (4.21)

Since $M$ is minimal, $\text{tr} A_\alpha = 0$, $\alpha = 1, \ldots, p$. Therefore substituting (4.18), (4.19) and (4.21) into (4.17), we obtain
\[ \int_M \{ \frac{1}{2} \| \mathcal{L}_U g \|^2 + (n + 1) - \text{tr} A_1^2 \} \ast 1 = 0. \] (4.22)

Thus we have the inequality (4.16). \[ \square \]

From Theorem A, we have immediately

Corollary 2. Let $M$ be as in Theorem 1. If the normal vector field $N_1$ is parallel with respect to the normal connection and
\[ \int_M \text{tr} A_1^2 \ast 1 = (n + 1) \text{Vol}(M), \]
then $M$ is locally a product of $M_1 \times M_2$ where $M_1$ and $M_2$ belong to some odd-dimensional spheres.

Proof. From the hypotheses and (4.22), we have $\| \mathcal{L}_U g \|^2 = 0$ and consequently $A_1 F = F A_1$ because of (4.20). Combining Theorem A and $A_1 F = F A_1$, we have the required result in Corollary 2. \[ \square \]
REFERENCES


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