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D.Q.M.을 이용한 I-단면 곡선보의 진동해석

Differential Quadrature Analysis for Vibration of Wide-Flange Curved Beams

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ABSTRACT

The differential quadrature method (D.Q.M.) is applied to computation of eigenvalues of small-amplitude free vibration for horizontally curved beams including a warping contribution. Fundamental frequencies are calculated for a single-span, curved, wide-flange beam with both ends simply supported or clamped, or simply supported-clamped end conditions. The results are compared with existing exact solutions and numerical solutions by other methods for cases in which they are available. The differential quadrature method gives good accuracy even when only a limited number of grid points is used.

국 문 요 약

I-단면 곡선보(curved beam)의 뒤틀림(warping)을 포함한 평면외(out-of-plane)의 자유진동을 해석하는데 differential quadrature method(D.Q.M.)을 이용하여 다양한 경계조건(boundary conditions)과 굽힘각(opening angles)에 따른 진동수(frequencies)를 계산하였다.

D.Q.M.의 결과는 해석적 해답(exact solution) 또는 다른 수치해석(Rayleigh-Ritz 또는 FEM) 결과와 비교하였으며, D.Q.M.은 적은 요소(grid points)를 사용하여 정확한 해석결과를 보여주었다.

1. Introduction

Owing to their importance in many fields of technology and engineering, the dynamic behavior of horizontally curved girders has been the subject of a large number of in-

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vestigations. Solutions of the relevant differential equations have traditionally been obtained by the standard finite difference method or finite element method. These techniques require a great deal of computation time as the number of discrete nodes becomes relatively large under conditions of complex geometry and loading. In many cases, the moderately accurate solution which can be calculated rapidly is desired at a few points in physical domain.

Culver[1] and Shore and Chaudhuri[2] studied the free vibration of horizontally curved beams using closed-form solutions. Tan and Shore[3] calculated the dynamic response of a single-span curved beam to moving loads. Chaudhuri and Shore[4] studied the free vibration of horizontally curved beams using the finite element method(FEM). Snyder and Wilson[5] calculated the free vibration frequencies of continuous horizontally curved beams using a nonexplicit closed-form solution of the partial differential equations of motion.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti[6]. This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. The objectives of the work are to apply the method to certain new problems, not previously reported in the literature. This method is used in the present work to analyze the free vibration behavior of circularly curved beams, specifically, of a single-span, horizontally curved, wide-flange beam including a warping contribution but neglecting the effects of rotatory inertia and transverse shearing deformation. The lowest frequencies are calculated for the member. The cross-sectional shape is assumed to be constant along the entire length of the member and doubly symmetric ; i.e., the shear center and centroid coincide. The member has both ends either simply supported or clamped, or has simply supported-clamped ends. Numerical results are compared with existing exact solutions and numerical solutions by the Rayleigh -Ritz method and the finite element method.

2. Governing Differential Equations

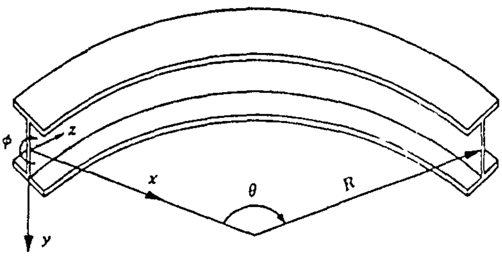

Fig. 1 Coordinate system for wide-flange curved beam

The differential equations governing a horizontally curved beam as shown in Fig. 1 can be written as (Culver[1])

$$\left(\frac{EI_w}{R^2}+EI_x\right)\frac{\partial^4 v}{\partial z^4}-\frac{GK_T}{R^2}\frac{\partial^2 v}{\partial z^2}+\frac{EI_w}{R}\frac{\partial^4\phi}{\partial z^4}-\frac{EI_x+GK_T}{R}\frac{\partial^2\phi}{\partial z^2}=mA\frac{\partial^2 v}{\partial t^2}\quad\cdots\cdots(1)$$

$$\frac{EI_w}{R}\frac{\partial^4 v}{\partial z^4}-\frac{EI_w+GK_T}{R}\frac{\partial^2 v}{\partial z^2}+EI_w\frac{\partial^4\phi}{\partial z^4}-GK_T\frac{\partial^2\phi}{\partial z^2}+\frac{EI_x}{R^2}\phi=mI_p\frac{\partial^2\phi}{\partial t^2}\quad\cdots\cdots(2)$$

where E is the modulus of elasticity, G is the shear modulus, I_p is the polar moment of inertia of the cross section about the shear center, I_x is the rectangular area moment of inertia about the x-axis (see Fig. 1), I_w is the warping constant, K_T is the Saint-Venant torsion constant, m is the density, R is the mean radius of curvature, v is the displacement of the shear center in the y-direction, and φ is the angle of twist of the beam cross section.

To find the corresponding free vibration frequencies, the following normal-mode solutions are assumed :

$$v(z,t) = V(z)\sin\omega t \quad \text{(3a)}$$

$$\phi(z,t) = \Phi(z)\sin\omega t \quad \text{(3b)}$$

Here, ω is the circular frequency of the member.

Replacing z by $R\theta$ and using (3a,b), one can rewrite (1) and (2) as

$$\left(\frac{EI_w}{R^2}+EI_x\right)\frac{V^{IV}}{R^4\theta_0^4}-\frac{GK_T}{R^2}\frac{V''}{R^2\theta_0^2}+\frac{EI_w}{R}\frac{\Phi^{IV}}{R^4\theta_0^4}-\frac{EI_x+GK_T}{R}\frac{\Phi''}{R^2\theta_0^2}-mA\omega^2V=0 \quad \text{(4)}$$

$$\frac{EI_w}{R}\frac{V^{IV}}{R^4\theta_0^4}-\frac{EI_w+GK_T}{R}\frac{V''}{R^2\theta_0^2}+EI_w\frac{\Phi^{IV}}{R^4\theta_0^4}-GK_T\frac{\Phi''}{R^2\theta_0^2}+\frac{EI_x}{R^2}\Phi-mI_p\omega^2\Phi=0 \quad \text{(5)}$$

where each prime denotes one differentiation with respect to the dimensionless distance coordinate $X=\theta/\theta_0$, in which θ_0 is the opening angle of the member and θ is the angle from left support to a generic point.

The following boundary conditions are taken for simply supported ends (Tan and Shore[7]) : (a) No vertical deflection ; (b) no torsional rotation ; (c) no bending moments ; and (d) the planes of the end cross section are free to warp.

The bending moment M_x and the warping normal stress σ_w of the beam can be written as (Christiano and Culver[8])

$$M_x = EI_x\left(\frac{\phi}{R}-\frac{d^2v}{dz^2}\right) \quad \text{(6)}$$

$$\sigma_w=\frac{B_w\Omega}{I_w}=-E\Omega\left(\frac{d^2\phi}{dz^2}+\frac{1}{R}\frac{d^2v}{dz^2}\right) \quad \text{(7)}$$

The warping stress is expressed in terms of the cross-sectional property Ω, the unit warping, and the bimoment $B_w(B_w=\int\sigma\Omega\, dA)$, which equals zero at simply supported ends.

For clamped ends, v, φ, dv/dz, and τ equal zero where τ represents the warping as defined by Vlasov[9]). It can be written as (Chaudhuri and Shore[10])

$$\tau(z)=-\left(\frac{1}{R}\frac{dv}{dz}+\frac{d\phi}{dz}\right) \quad \text{(8)}$$

The boundary conditions for both ends simply supported, both ends clamped, and for mixed simply supported-clamped ends are, respectively

$$v=\phi=v''=\phi''=0, \quad \text{at} \quad \theta=0,\theta_0$$

for both ends simply supported .... (9)

$$v=\phi=v'=\phi'=0, \quad \text{at} \quad \theta=0,\theta_0$$

for both ends clamped .... (10)

$$v=\phi=v''=\phi''=0, \quad \text{at} \quad \theta=0,$$

$$v=\phi=v'=\phi'=0, \quad \text{at} \quad \theta=\theta_0$$

for simply supported-clamped ends .. (11)

3. Differential Quadrature Method

The Differential Quadrature Method was introduced by Bellman and Casti[6]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al.[11]. The versatility of the D.Q.M. to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kukreti et al.[12] calculated the fundamental frequencies of tapered plates, and Farsa et al.[13] applied the method to analysis and detailed parametric evaluation of the fundamental frequencies of general anisotropic and laminated plates. In another development, the quadrature method was introduced in lubrication mechanics by Malik and Bert[14]. Kang and Bert[15] applied the method to the flexural-torsional buckling analysis of circular arches. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j)$$

$$\text{for} \quad i,\ j=1,2,\ldots,N \qquad (12)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x)=x^{k-1} \quad \text{for} \quad k=1,2,3,\ldots,N \qquad (13)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1}=(k-1)(k-2)\cdots(k-n)x_i^{k-n-1}$$

$$\text{for} \quad i,k=1,2,\ldots,N \qquad (14)$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming[16].

4. Application

The method of differential quadrature is applied here to the free vibration analysis of horizontally curved beams. The differential quadrature approximations of the governing equation and the boundary conditions are given below.

Applying the D.Q.M. to Eqs.(4) and (5) gives

$$\left(\frac{EI_w}{R^2}+EI_x\right)\frac{1}{R^4\theta_0^4}\sum_{j=1}^{N}D_{ij}V_j-\frac{GK_T}{R^2}\frac{1}{R^2\theta_0^2}\sum_{j=1}^{N}B_{ij}V_j+\frac{EI_w}{R}\frac{1}{R^4\theta_0^4}\sum_{j=1}^{N}D_{ij}\Phi_j-\frac{EI_x+GK_T}{R}\frac{1}{R^2\theta_0^2}\sum_{j=1}^{N}B_{ij}\Phi_j-mA\omega^2V_i=0, \quad j=1,2,3,\cdots,N \qquad (15)$$

$$\frac{EI_w}{R}\frac{1}{R^4\theta_0^4}\sum_{j=1}^{N}D_{ij}V_j-\frac{EI_x+GK_T}{R}\frac{1}{R^2\theta_0^2}\sum_{j=1}^{N}B_{ij}V_j+EI_w\frac{1}{R^4\theta_0^4}\sum_{j=1}^{N}D_{ij}\Phi_j-GK_T\frac{1}{R^2\theta_0^2}\sum_{j=1}^{N}B_{ij}\Phi_j+\frac{EI_x}{R^2}\Phi_i-mI_p\omega^2\Phi_i=0, \quad j=1,2,3,\cdots,N \qquad (16)$$

where B_{ij} and D_{ij} are the weighting coefficients for the second- and fourth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for simply supported ends, given by Eqs.(9), can be expressed in differential quadrature form as

$$V_1=0 \quad \text{at} \quad X=0 \qquad (17)$$

$$\Phi_1=0 \quad \text{at} \quad X=0 \qquad (18)$$

$$\sum_{j=1}^{N}B_{2j}V_j=0 \quad \text{at} \quad X=0+\delta \qquad (19)$$

$$\sum_{j=1}^{N}B_{2j}\Phi_j=0 \quad \text{at} \quad X=0+\delta \qquad (20)$$

$$\sum_{j=1}^{N} B_{(N-1)j} V_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots\cdots (21)$$

$$\sum_{j=1}^{N} B_{(N-1)j} \Phi_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots\cdots (22)$$

$$V_N = 0 \quad \text{at} \quad X=1 \quad \cdots\cdots (23)$$

$$\Phi_N = 0 \quad \text{at} \quad X=1 \quad \cdots\cdots (24)$$

The boundary conditions for clamped ends, given by Eqs.(10), can be expressed in differential quadrature form as

$$V_1 = 0 \quad \text{at} \quad X=0 \quad \cdots\cdots (25)$$

$$\Phi_1 = 0 \quad \text{at} \quad X=0 \quad \cdots\cdots (26)$$

$$\sum_{j=1}^{N} A_{2j} V_j = 0 \quad \text{at} \quad X=0+\delta \quad \cdots\cdots (27)$$

$$\sum_{j=1}^{N} A_{2j} \Phi_j = 0 \quad \text{at} \quad X=0+\delta \quad \cdots\cdots (28)$$

$$\sum_{j=1}^{N} A_{(N-1)j} V_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots\cdots (29)$$

$$\sum_{j=1}^{N} A_{(N-1)j} \Phi_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots\cdots (30)$$

$$V_N = 0 \quad \text{at} \quad X=1 \quad \cdots\cdots (31)$$

$$\Phi_N = 0 \quad \text{at} \quad X=1 \quad \cdots\cdots (32)$$

where A_{ij} are the weighting coefficients for the first-order derivative, and δ denotes a very small distance measured along the dimensionless axis from the boundary ends. This δ approach is used to apply more than one boundary condition at a given station.

Similarly, the boundary conditions for one simply supported and one clamped end, given by Eqs.(11), can be expressed in differential quadrature form as

$$V_1 = 0 \quad \text{at} \quad X=0 \quad \cdots\cdots (33)$$

$$\Phi_1 = 0 \quad \text{at} \quad X=0 \quad \cdots\cdots (34)$$

$$\sum_{j=1}^{N} B_{2j} V_j = 0 \quad \text{at} \quad X=0+\delta \quad \cdots\cdots (35)$$

$$\sum_{j=1}^{N} B_{2j} \Phi_j = 0 \quad \text{at} \quad X=0+\delta \quad \cdots\cdots (36)$$

$$\sum_{j=1}^{N} A_{(N-1)j} V_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots\cdots (37)$$

$$\sum_{j=1}^{N} A_{(N-1)j} \Phi_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots\cdots (38)$$

$$V_N = 0 \quad \text{at} \quad X=1 \quad \cdots\cdots (39)$$

$$\Phi_N = 0 \quad \text{at} \quad X=1 \quad \cdots\cdots (40)$$

Mixed boundaries can be easily accommodated by combining these equations. While most analytical methods use the rather laborious technique of superposition to arrive at solutions for mixed boundary problems, this approach of breaking the problem into several easy subproblems is not required in differential quadrature method. This set of equations together with the appropriate boundary conditions can be solved for the fundamental natural frequencies of the member.

5. Numerical Results and Comparisons

The fundamental natural frequencies of a horizontally curved beam are calculated by the differential quadrature method and are presented together with existing exact solutions and numerical solutions by the Rayleigh-Ritz method and the finite element method. The fundamental natural frequencies are evaluated for the case of a single-span, horizontally curved, wide-flange beam with both ends simply supported or clamped, or with mixed simply supported-clamped end conditions.

The first example considered here has a constant radius of curvature of 326.136 cm (128.4 in.) and a variety of opening angles ranging from 10° to 90°. Cross-sectional properties of the beam are :

A=92.9 cm^2(14.4 in.^{2}), I_x=11362 cm^4(273 in.^{4}), I_y=3817 cm^4(93 in.^{4}), I_w=555878 cm^6(2070 in.^{6}), and K_T=1470.84 cm^4(35.34 in.^{4}).

Values used for the elastic modulus, shear modulus, and density are :

E=200.1 GN/m^2(29000 ksi), G=77.3 GN/m^2 (11200 ksi), and m=0.786×10^{-5} N−sec^2/cm^4 (0.735×10^{-6} kip−sec^2/in.^{4}).

Tables 1 and 2 present the results of convergence studies relative to the number of grid points N and the δ parameter, respectively, with $\theta_0=90°$. Table 1 shows that the accuracy of the numerical solution increases with increasing N and passes through a maximum. Then, numerical instabilities arise if N becomes too large. Table 2 shows the sensitivity of the numerical solution to the choice of δ. The optimal value for δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too small. The remainder of the numerical results are computed with thirteen discrete points along the dimensionless X-axis and $\delta=1\times10^{-6}$.

Table 1 Fundamental frequency of free vibration, ω, of curved beams with both ends simply supported for a range of grid points, $\theta_0=90°$; warping and torsional inertia included

Culver[1] (Exact)	Number of grid points			
ω, radians per second	7	9	11	13
64.615	68.950	64.552	64.629	64.628

Table 2 Fundamental frequency of free vibration, ω, of curved beams with both ends simply supported for a range of δ, $\theta_0=90°$; warping and torsional inertia included

Culver[1] (Exact)	δ				
ω, radians per second	1×10^{-2}	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
64.615	76.972	65.946	64.749	64.628	64.616

Shore and Chaudhuri[2] determined the natural frequencies of the member neglecting the torsional inertia term, $mI_p\omega^2\varphi$, in Eq.(4). In Table 3, the natural frequencies determined by the differential quadrature method are compared with the exact solutions by Shore and Chaudhuri[2] for the case of simply supported ends.

In Table 4, the exact solutions by Culver[1] are compared with those by the D.Q.M. including or neglecting warping deformation for the case of simply supported ends.

From Table 4, the natural frequencies of the member including warping deformations are higher than those of the member neglecting warping deformations. From Tables 3 and 4, the natural frequencies of the member neglecting torsional inertia are higher than those of the member including torsional inertia.

Culver[1] determined the natural frequencies of the member using the Rayleigh-Ritz method

Table 3 Fundamental frequency of free vibration, ω, of curved beams with both ends simply supported ; warping included and torsional inertia neglected

θ_0, degrees	ω, radians per second	
	Shore and Chaudhuri[2] (Exact)	D.Q.M
10°	16815	16795
20°	3928.1	3913.3
30°	1542.2	1536.2
40°	745.96	743.81
50°	406.78	406.01
60°	240.21	239.91
70°	150.12	149.90
80°	97.526	97.475
90°	65.219	65.195

Table 4 Fundamental frequency of free vibration, ω, of curved beams with both ends simply supported ; with and without warping

θ_0, degrees	ω, radians per second			
	Culver[1] (Exact)	D.Q.M.	Culver[1] (Exact), $I_w=0$	D.Q.M. $I_w=0$
10°	10615	10614	5340.2	5339.6
20°	3130.1	3129.9	2460.8	2460.0
30°	1361.1	1361.1	1241.4	1241.3
40°	690.63	690.61	655.10	655.09
50°	387.50	387.50	373.35	373.35
60°	232.87	232.87	226.35	226.35
70°	147.01	147.01	143.74	143.74
80°	96.214	96.215	94.473	94.473
90°	64.615	64.616	63.648	63.648

Table 5 Fundamental frequency of free vibration, ω, of curved beams with both ends clamped ; warping and torsional inertia included

θ_0, degrees	ω, radians per second	
	Culver[1] (Rayleigh-Ritz solution)	D.Q.M.
10°	21875	21871
20°	6096.9	6091.3
30°	3131.8	3126.2
40°	1965.0	1962.8
50°	1308.8	1305.7
60°	916.58	909.92
70°	762.76	662.23
80°	513.23	499.09
90°	403.88	386.55

Table 6 Fundamental frequency of free vibration, ω, of curved beams with one simply supported and one clamped end ; warping and torsional inertia included

θ_0, degrees	ω, radians per second	
	Culver[1] (Rayleigh-Ritz solution)	D.Q.M.
10°	15588	15585
20°	4576.9	4571.1
30°	2283.7	2281.2
40°	1309.1	1305.5
50°	822.84	814.80
60°	556.81	543.87
70°	398.88	381.37
80°	298.71	277.35
90°	231.69	207.32

for the cases of clamped ends and mixed simply supported-clamped ends. The results are summarized in Tables 5 and 6. From Tables 4 and 5, the natural frequencies of the member with clamped ends are much higher than those of the member with simply supported ends. Tables 5 and 6 show that the natural frequencies of the member with clamped ends are higher than those of the member with simply supported-clamped ends. From Tables 5 and 6, the numerical results by the D.Q.M. are lower than those by the Rayleigh-Ritz method, and the difference of the numerical results between the two methods decreases as the opening angle of the member decreases.

Chaudhuri and Shore[4] determined the natural frequencies of the following example using the finite element method (FEM) for the cases of simply supported ends neglecting warping deformation and torsional inertia. This example considered here has a constant radius of curvature of 254 cm(100 in.) with $\theta_0 = 75°$. Cross-sectional properties of the beam are :

$A = 327\ cm^2\ (50.0\ in.^2)$, $I_x = 17341.8\ cm^4\ (416.67\ in.^4)$, $I_y = 4335.43\ cm^4\ (104.167\ in.^4)$, and $K_T = 11913.73\ cm^4\ (286.25\ in.^4)$.

Values used for the elastic modulus, shear modulus, and density are :

$E = 207000\ GN/m^2 (30.0 \times 10^6\ ksi)$, $G = 79615\ GN/m^2\ (11.54 \times 10^6\ ksi)$, and $m = 7.865 \times 10^{-2}\ N-sec^2/cm^4\ (7.35 \times 10^{-4}\ lb-sec^2/in.^4)$.

In Table 7, the natural frequencies determined by the D.Q.M. are compared with those by the FEM. Table 7 shows that the numerical results by the D.Q.M. are more accurate than those by the FEM (91 grid points). As can be seen, the numerical results by the differential quadrature method show excellent agreement with the exact solutions, the Rayleigh-Ritz solutions, and the finite element solutions.

Table 7 Fundamental frequency of free vibration, ω, of curved beams with both ends simply supported ; warping and torsional inertia neglected

θ_0, degrees	ω, radians per second		
	Shore and Chaudhuri[2] (Exact)	D.Q.M.	Chaudhuri and Shore[4] (Finite element solution)
75°	215.66	215.57	214.24

6. Conclusions

The differential quadrature method was

used to compute the eigenvalues of free vibration of horizontally curved beams including warping deformations. The present method gives results which agree very well with the exact ones and with numerical solutions by other methods for the cases treated while requiring only a limited number of grid points.

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