한국퍼지 및 지능 시스템학회 논문지 1998, Vol. 8, No. 6. (1998) 21 ~ 26

Some Characterizations of *TL*-subgroups

Han-Doo Kim, Dong-Seog Kim* and Jae-Gyeom Kim*

*Dept. of Mathematics, Inje University and *Dept. of Mathematics, Kyungsung University*

ABSTRACT

In this paper, we show that if a *TL*-subgroup can be written as the intersection of all its minimal *TL*-*p*-subgroups then some properties of the *TL*-subgroup characterize the properties of all its minimal *TL*-*p*-subgroups and investigate the properties of the join of a directed family of *TL*-subgroups.

1. Introduction

Rosenfeld[9] introduced the concept of fuzzy subgroups. Following these ideas, many authors are engaged in generalizing various notions of group theory in the fuzzy setting. In particular, the notion of fuzzy orders of the elements of a group relative to a fuzzy group and the notion of fuzzy orders of fuzzy subgroups have been introduced[4] and developed[3, 4,6,8] and conditions for a fuzzy subgroup to be written as the intersection of its minimal fuzzy *p*-subgroups have been investigated[2,3,5]. Recently the concept of *TL*-subgroups that is an extension of the concept of fuzzy subgroups has been introduced and studied[11] and the notion of *TL*-orders of the elements of a group relative to a *TL*-subgroup and the notion of *TL*-orders of *TL*-subgroups that are extensions of the notion of fuzzy orders of the elements and the notion of fuzzy orders of fuzzy subgroups, respectively, were introduced [7] and developed[2,5]. In this paper, we show that if a *TL*-subgroup can be written as the intersection of all its minimal *TL*-*p*-subgroups then some properties of the *TL*-subgroup characterize the properties of all its minimal *TL*-*p*-subgroups and investigate the properties of the join of a directed family of *TL*-subgroups.

Throughout this paper, we let L denote a complete lattice that contains at least two distinct elements. The meet, join, and partial ordering will be written as $\wedge$, $\vee$, and $\leq$, respectively. We also write 1 for the greatest element of L.

2. Preliminaries

We recall basic definitions and investigate some properties that are relevant for this paper.

Definition 2.1. [10] A binary operation T on L is called a *t*-norm if it satisfies the following:

(1) $(aTb)Tc = aT(bTc)$ for all $a, b, c \in L$.

(2) $aTb = bTa$ for all $a, b \in L$.

(3) If $b \leq c$, then $aTb \leq aTc$ for all $a, b, c \in L$.

(4) $aT1 = a$ for all $a \in L$.

We will write the identity element of a group G by e and the order of x in G by $O(x)$. And we let T denote a *t*-norm on L.

Definition 2.2. [11] An L-subset μ of a group G, i. e., a function μ from G to L, is called a *TL*-subgroup of G if it satisfies the following:

(1) $\mu(e)=1$.

(2) $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

(3) $\mu(xy) \geq \mu(x)T\mu(y)$ for all $x, y \in G$.

Note that the concept of a *TL*-subgroup is an extension of the concept of a fuzzy subgroup.

Definition 2.3. [7] Let μ be a *TL*-subgroup of a group G. For a given $x \in G$, the least positive integer n such that $\mu(x^n)=1$ is said to be the (*TL*-) order of x with respect to μ (briefly, $O_\mu(x)$). If no such n exists, x is said to have infinite (*TL*-) order with respect to μ.

Note that the notion of the *TL*-order of x is a generalization of the notion of the fuzzy order of x.

*This work was supported by Inje Research and Scholarship Foundation in 1997.

Proposition 2.4. [7] Let μ be a TL-subgroup of a group G. For $x \in G$, if $\mu(x^m)=1$ for some integer m, then $O_\mu(x)$ divides m.

Definition 2.5. [7] Let μ be a TL-subgroup of a group G. For a prime p, μ is called a TL-p-subgroup of G if $O_\mu(x)$ is a power of p for every $x \in G$.

Let μ be a TL-subgroup of a group G. If there exists a minimal TL-p-subgroup of G containing μ, then it is unique because the intersection of TL-p-subgroups of G is obviously a TL-p-subgroup. We will call it by the least TL-p-subgroup of G containing μ and denote it by $\mu_{(p)}$. Note that for every prime p, $\mu_{(p)}$ does not exist in general even if $T=\wedge$ and $L=[0, 1]$[4].

Note that the homomorphic images and the homomorphic preimages of TL-subgroups are TL-subgroups[11]. And note that the homomorphic images and the homomorphic preimages of TL-p-subgroups are TL-p-subgroups[5].

Definition 2.6. [5] Let μ be a TL-subgroup of an Abelian group G. μ is said to be torsion if $O_\mu(x)$ is finite for all $x \in G$.

Proposition 2.7. Let μ be a TL-subgroup of a group G such that $G_\mu=\{x \in G \mid \mu(x)=1\}$ is a normal subgroup of G. Then μ is TL-p-subgroup if and only if G/G_μ is a p-group.

Proof. Suppose G/G_μ is not a p-group. Then $x^{p^t} \notin G_\mu$ for all integers t with $t \geq 0$ for some $x \in G$. Thus $\mu(x^{p^t})<1$ for all integers t with $t \geq 0$. Thus $O_\mu(x)$ is not a power of p. Therefore μ is not a TL-p-subgroup of a group G. Suppose μ is not a TL-p-subgroup of a group G. Then $O_\mu(x)$ is not a power of p for some $x \in G$. Thus $x^{p^t} \notin G_\mu$ for all integers t with $t \geq 0$. Therefore G/G_μ is not a p-group.

Note that if μ is a TL-subgroup of a group G with $T=\wedge$, then $\mu(x^n) \geq \mu(x)$ for all $x \in G$ and for all integers n.

Proposition 2.8. Let μ be a TL-subgroup of a group G with $T=\wedge$. Let $x, y \in G$.

(1) If $\mu(x)<\mu(y)$, then $\mu(xy) \wedge \mu(y)=\mu(x)=\mu(yx) \wedge \mu(y)$.

(2) If $\mu(x) \leq \mu(y)$, then $\mu(x) \leq \mu(xy)$ and $\mu(x) \leq \mu(yx)$.

Proof. (1) Let $\mu(x)<\mu(y)$. Then we have $\mu(xy) \geq \mu(x) \wedge \mu(y)=\mu(x)$. So $\mu(x)=\mu(x(yy^{-1}))=\mu((xy)y^{-1}) \geq \mu(xy) \wedge \mu(y^{-1})=\mu(xy) \wedge \mu(y) \geq \mu(x)$.

Thus $\mu(xy) \wedge \mu(y)=\mu(x)$.

(2) If $\mu(x) \leq \mu(y)$, then $\mu(x)=\mu(x) \wedge \mu(y) \leq \mu(xy)$ and $\mu(x)=\mu(y) \wedge \mu(x) \leq \mu(yx)$.

Let μ be a TL-subgroup of a group G satisfying the following condition: If $\mu(x)$ and $\mu(y)$ are comparable where $x, y \in G$, then $\mu(xy)=\mu(x) \wedge \mu(y)$. A TL-subgroup μ of a group G satisfying this condition is said to have the property (E).

Theorem 2.9. Let μ be a TL-subgroup of a group G with $T=\wedge$ and p a prime. And let μ have the property (E). Suppose that for every $x \in G$, min $\{n \in N \mid \mu(x)<\mu(x^n)\}$ is a power of p, whenever this minimum exists. And let n, m, t be positive integers such that $n=p^t m$ and $(p, m)=1$. Then $\mu(x^n)=\mu(x^{p^t})$ *for all* $x \in G$.

Proof. Let $x \in G$. Since $n=p^t m$ and $T=\wedge$, $\mu(x^{p^t}) \leq \mu(x^n)$. Let $\mu(x^{p^t})<\mu(x^n)$. Set $y=x^{p^t}$. Then $\mu(y)<\mu(y^m)$. By hypothesis, there is a natural number s such that $p^s=\min \{k \in N \mid \mu(y)<\mu(y^k) \leq m$, and so $\mu(y^j)=\mu(y)$ for $j=1, 2, \cdots, p^s-1$. By the Euclidean algorithm, there are integers q and r such that $m=p^s q+r$ and $0 \leq r<p^s$. Here $r \neq 0$, since $(p, m)=1$. So, $\mu(y)<\mu(y^{p^s}) \leq \mu(y^{p^s q})$ since $T=\wedge$ and $\mu(y)=\mu(y^r)$. Thus $\mu(y^m)=\mu(y^{p^s q} y^r)=\mu(y^r)=\mu(y)$ since μ has the property (E). This is a contradiction, and the theorem is proved.

Theorem 2.10. [7] Let μ be a TL-subgroup of a group G. For $x \in G$, if $O_\mu(x)=mn$ with $(m, n)=1$, then there exist x_1 and x_2 in G such that $x=x_1x_2=x_2x_1$, $O_\mu(x_1)=m$, and $O_\mu(x_2)=n$. Furthermore such an expression for x is unique in the sense of TL-grades, i.e., if (x_1, x_2) and (y_1, y_2) are such pairs, then $\mu(x_1)=\mu(y_1)$ and $\mu(x_2)=\mu(y_2)$.

Let μ be a TL-subgroup of a group G with $T=\wedge$ such that $O_\mu(x)$ is finite for all $x \in G$. For every prime p, define a L-subset μ_p of G by $\mu_p(x)=\mu(x_2)$ where $x=x_1x_2$ is an expression for x with $O_\mu(x)=mp^t$, $(p, m)=1$, $O_\mu(x_1)=m$, and $O_\mu(x_2)=p^t$. Then μ_p is well-defined by Theorem 2.10.

Proposition 2.11. [5] Let μ be a torsion TL-subgroup of an Abelian group G with $T=\wedge$. For every prime p, μ_p is the least TL-p-subgroup of G containing μ i.e., $\mu_p=\mu_{(p)}$.

3. Characterizations of TL-subgroups

Some properties of the intersection of TL-subgroups can be characterized by the properties of the TL-subgroups, but the converse does not hold in general, i.e., the properties of the intersection cannot characterize the properties of the TL-subgroups.

Now we have a natural question; for what conditions does the converse hold? For some properties, the answer is affirmative in the case that $\mu = \cap \mu_{(p)}$ holds. We are concerned with this problem in this section.

Definition 3.1. [11] A TL-subgroup μ of a group G is called a normal TL-subgroup of G if it satisfies the following equivalent conditions:

(1) $\mu(yx)=\mu(xy)$ for all $x, y\in G$.

(2) $\mu(xyx^{-1})=\mu(y)$ for all $x,y\in G$.

Lemma 3.2. Let μ be a TL-subgroup of a group G. For $g\in G$, let $i_g: G\rightarrow G$ be the automorphism from G into G defined by $i_g(x)=g^{-1}xg$ for all $x\in G$. And let μ be normal. For a given prime p, if there exists $\mu_{(p)}$, then $(i_g)^{-1}(\mu_{(p)})\supseteq\mu_{(p)}$.

Proof. For each $g\in G$, let $i_g: G\rightarrow G$ be the function from G into G defined by $i_g(x)=g^{-1}xg$ for all $x\in G$. Then i_g is an automorphism of G. Since the homomorphic preimage of TL-p-subgroup is TL-p-subgroup, $(i_g)^{-1}(\mu_{(p)})$ is a TL-p-subgroup. And let μ be normal. Since μ is normal, we have $\mu(x)=\mu(g^{-1}xg)\leq \mu_{(p)}(g^{-1}xg)=\mu_{(p)}(i_g(x))=(i_g)^{-1}(\mu_{(p)})(x)$ for all $x\in G$, and so $(i_g)^{-1}(\mu_{(p)})\supseteq\mu$. Since $\mu_{(p)}$ is the minimal TL-p-subgroup containing μ, $(i_g)^{-1}(\mu_{(p)})\supseteq\mu_{(p)}$.

We have a proposition from Lemma 3.2.

Proposition 3.3. Let μ be a TL-subgroup of a group G such that $\mu=\cap\mu_{(p)}$. If μ is normal, then $\mu_{(p)}$ is normal for all primes p.

Proof. $\mu_{(p)}(g^{-1}xg)$

$=((i_g)^{-1}(\mu_{(p)}))(x))$

$\geq \mu_{(p)}(x)$ by Lemma 3.2

$=\mu_{(p)}(gg^{-1}xgg^{-1})$

$=\mu_{(p)}(i_{g^{-1}}(g^{-1}xg))$

$=((i_{g^{-1}})^{-1}(\mu_{(p)}))(g^{-1}xg)$

$\geq \mu_{(p)}(g^{-1}xg)$ by Lemma 3.2

for all g, $x\in G$. So $\mu_{(p)}$ is normal.

It is clear that the fact that a TL-subgroup μ is normal does not imply that all TL-subgroups μ_i are normal where $\mu=\cap\mu_i$ in general.

Definition 3.4. [10] A TL-subgroup μ of a group G is called a characteristic TL-subgroup of G if the preimage $f^{1}(\mu)$ of μ under f is equal to μ for all automorphisms f of G.

Lemma 3.5. Let μ be a TL-subgroup of a group G. Let f be an automorphism of G. And let μ be characteristic. For a given prime p, if there exists $\mu_{(p)}$, then $f^{1}(\mu_{(p)})\supseteq\mu_{(p)}$.

Proof. Let f be an automorphism of G. And let μ be characteristic. Since $\mu\subseteq\mu_{(p)}$, $\mu=f^{-1}(\mu)\subseteq f^{-1}(\mu_{(p)})$. Since the homomorphic preimage of TL-p-subgroup is TL-p-subgroup, $f^{-1}(\mu_{(p)})$ is a TL-p-subgroup. Since $\mu_{(p)}$ is the minimal TL-p-subgroup containing μ, $\mu_{(p)}\subseteq f^{-1}(\mu_{(p)})$.

We have a proposition from Lemma 3.5.

Proposition 3.6. Let μ be a TL-subgroup of a group G such that $\mu=\cap\mu_{(p)}$. If μ is characteristic, then $\mu_{(p)}$ is characteristic for all primes p.

Proof. By Lemma 3.5, we have $\mu_{(p)}\subseteq f^{-1}(\mu_{(p)})$. Also by Lemma 3.5,

$$\begin{aligned}\mu_{(p)}(x)&=\mu_{(p)}(f^{-1}(f(x)))\\&=(f^{-1})^{-1}\mu_{(p)}(f(x))\\&\geq\mu_{(p)}(f(x))\\&=f^{-1}(\mu_{(p)})(x) \text{ for all } x\in G.\end{aligned}$$

Thus we have $f^{-1}(\mu_{(p)})\subseteq\mu_{(p)}$. Therefore, $f^{-1}(\mu_{(p)})=\mu_{(p)}$ for all automorphisms f of G.

Lemma 3.7. [7] Let μ be a TL-subgroup of a group G with $T=\wedge$. Let $O_\mu(x)=m$ where $x\in G$. If n is an integer with $(m, n)=1$, then $\mu(x^n)=\mu(x)$.

Theorem 3.8. Let μ be a TL-subgroup of a group G with $T=\wedge$ such that $\mu=\cap\mu_{(p)}$. For $x\in G$ with $O_\mu(x)=p^t m$, where p is a prime and $(p, m)=1$, let x_1 and x_2 be such that $x=x_1x_2=x_2x_1$, $O_\mu(x_1)=p^t$ and $O_\mu(x_2)=m$. Then the following hold:

(1) $\mu_{(p)}(x)=\mu_{(p)}(x_1)=\mu(x_1)$

(2) $O_{\mu_{(p)}}(x)=O_{\mu_{(p)}}(x_1)=p^t$

Proof. Let $\mu=\cap\mu_{(p)}$. And let q be a prime different from p. Since $(O_{\mu_{(q)}}(x_1), p^t)=1$, $\mu_{(q)}(x_1^{p^t})=\mu_{(q)}(x_1)$ by

Lemma 3.7. Since $\mu_{(q)}(x_1^{p^t})\geq\mu(x_1^{p^t})=1$, $\mu_{(q)}(x_1)=1$. Since $(O_{\mu_{(p)}}(x_2), m)=1$, $\mu_{(p)}(x_2^m)=\mu_{(p)}(x_2)$ by Lemma 3.7. Since $\mu_{(p)}(x_2^m)\geq\mu(x_2^m)=1$, $\mu_{(p)}(x_2)=1$. Thus $\mu_{(p)}(x)=\mu_{(p)}(x_1x_2)=\mu_{(p)}(x_1)=\mu(x_1)$ since $T=\wedge$ and $\mu=\cap\mu_{(p)}$. And $\mu_{(p)}(x_1^n)=\mu(x_1^n)$ for all integers n, because $\mu_{(q)}(x_1^n)\geq\mu_{(q)}(x_1)=1$ and $\mu=\cap\mu_{(p)}$. Thus $\mu_{(p)}(x^n)=\mu_{(p)}(x_1^nx_2^n)=\mu_{(p)}(x_1^n)=\mu(x_1^n)$ for all integers n, since $T=\wedge$, and so $O_{\mu_{(p)}}(x)=O_{\mu_{(p)}}(x_1)=O_\mu(x_1)=p^t$.

Definition 3.9. [11] A TL-subgroup μ of a group G is called a fully invariant TL-subgroup of G if the image $f(\mu)$ of μ under f is contained in μ for all endomorphisms f of G.

Clearly, a fully invariant TL-subgroup is characteristic.

Proposition 3.10. Let μ be a TL-subgroup of a group G with $T=\wedge$ such that $\mu=\cap\mu_{(p)}$ and $O_\mu(x)$ is finite for all $x\in G$. If μ is fully invariant, then $\mu_{(p)}$ is fully invariant for all primes p.

Proof. Let $x\in G$ such that $O_\mu(x)=p^tm$ and $(p, m)=1$. And let x_1 and x_2 be such that $x=x_1x_2=x_2x_1$, $O_\mu(x_1)=p^t$, and $O_\mu(x_2)=m$. Since $\mu=\cap\mu_{(p)}$, $\mu_{(p)}(x)=\mu_{(p)}(x_1)=\mu(x_1)$ by Theorem 3.8. Now let μ be fully invariant and f an endomorphism of G. Since $O_{u_{(p)}}(f(x_2))$ is a power of p, $\mu_{(p)}(f(x_2))=\mu_{(p)}(f(x_2)^m)$ by Lemma 3.7. Also, since $f(\mu)\subseteq\mu$, we have

$$\begin{aligned}\mu(f(x))&\geq f(\mu)(f(x))\\&=\sup\{\mu(z)\mid f(z)=f(x)\}\\&\geq\mu(x)\text{ for every }x\in G.\end{aligned}$$

Thus we have

$$\begin{aligned}\mu_{(p)}(f(x_2))&=\mu_{(p)}(f(x_2)^m)\\&\geq\mu(f(x_2)^m)=\mu(f(x_2^m))\geq\mu(x_2^m)=1,\end{aligned}$$

and so $\mu_{(p)}(f(x_2))=1$.

$$\begin{aligned}\text{Since }\mu_{(p)}(f(x))&=\mu_{(p)}(f(x_1x_2))\\&=\mu_{(p)}(f(x_1)f(x_2))\\&=\mu_{(p)}(f(x_1))\\&\geq\mu(f(x_1))\\&\geq\mu(x_1)\\&=\mu_{(p)}(x_1)\\&=\mu_{(p)}(x),\end{aligned}$$

we have

$$\begin{aligned}f(\mu_{(p)})(z)&=\sup\{\mu_{(p)}(x)\mid f(x)=z\}\\&\leq\sup\{\mu_{(p)}(f(x))\mid f(x)=z\}\\&=\mu_{(p)}(z)\text{ for every }z\in G.\end{aligned}$$

Hence $f(\mu_{(p)})\subseteq\mu_{(p)}$. So $\mu_{(p)}$ is fully invariant.

Definition 3.11. [2] A TL-subgroup μ of a finite group G is said to have the property (*) if $O_\mu(x)=O_\mu(y)$ implies $\mu(x)=\mu(y)$ for all $x, y\in G$.

Lemma 3.12. [7] Let μ be a TL-subgroup of a group G. If $\mu(x)=1$, then $\mu(xy)=\mu(y)=\mu(yx)$.

Proposition 3.13. Let μ be a TL-subgroup of a group G with $T=\wedge$ such that $\mu=\cap\mu_{(p)}$ and $O_\mu(x)$ is finite for all $x\in G$. Then μ has the property (*) if and only if $\mu_{(p)}$ has the property (*) for all primes p.

Proof. Let $x, y\in G$ such that $O_{\mu_{(p)}}(x)=O_{\mu_{(p)}}(y)=p^t$. Let $O_\mu(x)=p^lm$ with $(p, m)=1$. And let $x_1, x_2\in G$ be such that $x=x_1x_2=x_2x_1$, $O_\mu(x_1)=p^l$ and $O_\mu(x_2)=m$. Let q be a prime different from p. Since $(O_{\mu_{(q)}}(x_1), p^l)=1$, $\mu_{(q)}(x_1^{p^l})=\mu_{(q)}(x_1)$ by Lemma 3.7. Since $\mu_{(q)}(x_1^{p^l})\geq\mu(x_1^{p^l})=1$, $\mu_{(q)}(x_1)=1$. Since $(O_{\mu_{(p)}}(x_2), m)=1$, $\mu_{(p)}(x_2^m)=\mu_{(p)}(x_2)$ by Lemma 3.7. Since $\mu_{(p)}(x_2^m)\geq\mu(x_2^m)=1$, $\mu_{(p)}(x_2)=1$. Then we have $\mu_{(p)}(x)=\mu_{(p)}(x_1x_2)=\mu_{(p)}(x_1)=\mu(x_1)$, by Lemma 3.12 and $\mu=\cap\mu_{(p)}$. Since $\mu=\cap\mu_{(p)}$ and $\mu_{(q)}(x_1^n)\geq\mu_{(q)}(x_1)=1$, then we have $\mu_{(p)}(x_1^n)=\mu(x_1^n)$ for all integers n. And $\mu_{(p)}(x^n)=\mu_{(p)}(x_1^nx_2^n)=\mu_{(p)}(x_1^n)=\mu(x_1^n)$ for all integers n by Lemma 3.12, and so $p^t=O_{\mu_{(p)}}(x)=O_{\mu_{(p)}}(x_1)=O_\mu(x_1)=p^l$. Thus $l=t$. Thus we have $O_\mu(x)=p^tm$, $O_\mu(y)=p^tn$ with $(p, m)=(p, n)=1$. Now let x_1 and x_2 be such that $x=x_1x_2=x_2x_1$, $O_\mu(x_1)=p^t$, and $O_\mu(x_2)=m$. And let y_1 and y_2 be such that $y=y_1y_2=y_2y_1$, $O_\mu(y_1)=p^t$, and $O_\mu(y_2)=n$. Then $\mu_{(p)}(x)=\mu_{(p)}(x_1)=\mu(x_1)$ and $\mu_{(p)}(y)=\mu_{(p)}(y_1)=\mu(y_1)$ by Theorem 3.8. So if μ has the property (*) then $\mu_{(p)}$ has the property (*).

For the converse, let $O_\mu(x)=O_\mu(y)=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ where the p_i's are distinct primes. By Theorem 3.8, $O_{\mu_{(p_i)}}(x)=O_{\mu_{(p_i)}}(y)=p_i^{\alpha_i}$ for all $i\in\{1, 2, \cdots, r\}$. Since every $\mu_{(p_i)}$ has the property (*), $\mu_{(p_i)}(x)=\mu_{(p_i)}(y)$ for all $i\in\{1, 2, \cdots, r\}$. Let $q\notin\{p_1, \cdots, p_r\}$ and q be a prime. Since $(O_{\mu_{(q)}}(x), p_1^{\alpha_1}\cdots p_r^{\alpha_r})=1$, $\mu_{(q)}(x^{p_1^{\alpha_1}\cdots p_r^{\alpha_r}})=\mu_{(q)}(x)$. Since $\mu_{(q)}(x^{p_1^{\alpha_1}\cdots p_r^{\alpha_r}})\geq\mu(x^{p_1^{\alpha_1}\cdots p_r^{\alpha_r}})=1$, $\mu_{(q)}(x)=1$. Similarly $\mu_{(q)}(y)=1$. Thus $\mu_{(p)}(x)=\mu_{(p)}(y)$ for all primes p. Thus we have $\mu(x)=\mu(y)$.

Definition 3.14. [9] A *TL*-subgroup μ of a group G is called a generalized characteristic *TL*-subgroup of G if $O(x)=O(y)$ implies $\mu(x)=\mu(y)$ for all $x, y\in G$.

Proposition 3.15. Let μ be a *TL*-subgroup of a finite group G with $T=\wedge$ such that $\mu=\cap\mu_{(p)}$. If μ is a generalized characteristic *TL*-subgroup, then $\mu_{(p)}$ is a generalized characteristic *TL*-subgroup for all primes p.

Proof. Let $x, y\in G$ such that $O(x)=O(y)=p^t m$ and $(p, m)=1$. And let x_1 and x_2 be such that $x=x_1x_2=x_2x_1$, $O(x_1)=p^t$, and $O(x_2)=m$. Since $(O_{\mu_{(p)}}(x_2), m)=1$ and $O_{\mu_{(p)}}(x_2)$ divides $O(x_2)$, $O_{\mu_{(p)}}(x_2)$ divides m. Then we have $O_{\mu_{(p)}}(x_2)=1$ i.e., $\mu_{(p)}(x_2)=1$. Let q be a prime different from p. Since $O_{\mu_{(q)}}(x_1)$ is a power of q, $\mu_{(q)}(x_1^{p^t})=\mu_{(q)}(x_1)$ by Lemma 3.7. Since $\mu_{(q)}(x_1^{p^t})\geq\mu(x_1^{p^t})=1$, $\mu_{(q)}(x_1)=1$. Since $\mu=\cap\mu_{(p)}$, $\mu(x_1)=\mu_{(p)}(x_1)$. Thus $\mu_{(p)}(x)=\mu_{(p)}(x_1x_2)=\mu_{(p)}(x_1)=\mu(x_1)$ by Lemma 3.12.

Similarly, there is $y_1\in G$ such that $O(y_1)=p^t$ and $\mu_{(p)}(y)=\mu(y_1)$. So if μ is a generalized characteristic *TL*-subgroup, then $\mu_{(p)}$ is a generalized characteristic *TL*-subgroup.

4. Some Properties of a Directed Family of *TL*-subgroups

In this section, we study some properties of directed subfamily of *TL*-subgroups. We let L^G denote the set of all L-subsets of G.

Definition 4.1. [11] A family $\{\mu_i\,|\,\mu_i\in L^G, i\in I\}$ is said to be directed, where I is a nonempty index set if it satisfies the following conditions:

(1) For each $x\in G$, there exists an $i_x\in I$ such that $\mu_{i_x}(x)=\bigvee_{i\in I}\mu_i(x)$;

(2) For each pair $(i_1, i_2)\in I\times I$, there exists an $i_3\in I$ such that $\mu_{i_1}\vee\mu_{i_2}\leq\mu_{i_3}$.

It follows from this definition that if $\{\mu_i\,|\,\mu_i\in L^G, i\in I\}$ is a nonempty finite directed family of L^G, then $\bigvee_{i\in I}\mu_i=\mu_j$ for some $j\in I$.

Proposition 4.2. [1] Let $\{\mu_i\,|\,\mu_i\in L^G, i\in I\}$ be a directed family of L^G, where I is a nonempty index set. If μ_i is a *TL*-subgroup of G for each $i\in I$, then $\bigvee_{i\in I}\mu_i$ is a *TL*-subgroup of G.

Definition 4.3. [9] A *TL*-subgroup μ of a group G is said to have the property (B) if $\mu(x^n)\geq\mu(x)$ for all $x\in G$ and for all integers n.

Proposition 4.4. Let $\{\mu_i\,|\,\mu_i\in L^G, i\in I\}$ be a directed family of *TL*-subgroups of G, where I is a nonempty index set. If μ_i has the property (B) for each $i\in I$, then $\bigvee_{i\in I}\mu_i$ has the property (B).

Proof. Suppose that μ_i has the property (B) for each $i\in I$. Let $x\in G$. Then we have

$$\begin{aligned}\bigvee_{i\in I}\mu_i(x^n)&=\sup\{\mu_i(x^n)\,|\,i\in I\}\\&\geq\sup\{\mu_i(x)\,|\,i\in I\}\\&=\bigvee_{i\in I}\mu_i(x)\end{aligned}$$

for all integers n. So $\bigvee_{i\in I}\mu_i$ has the property (B).

Proposition 4.5. Let $\{\mu_i\,|\,\mu_i\in L^G, i\in I\}$ be a directed family of *TL*-subgroups of G, where I is a nonempty index set. Let $x\in G$ and $O_{\mu_i}(x)=a_i$ for all $i\in I$. And let $\{a_i\,|\,i\in I\}$ be finite. Then (1) $O_{\bigvee_{i\in I}\mu_i}(x)$ is a common divisor of $\{a_i\,|\,i\in I\}$.

(2) $O_{\bigvee_{i\in I}\mu_i}(x)$ is a divisor of $\Pi_{i\in I}a_i$, where $\Pi_{i\in I}a_i$ is the product of the elements in the set $\{a_i\,|\,i\in I\}$.

Proof. (1) For each $i\in I$, since $\bigvee_{i\in I}\mu_i(x^{a_i})\geq\mu_i(x^{a_i})=1$, $O_{\bigvee_{i\in I}\mu_i}(x)$ is a divisor of a_i by Proposition 2.4.

(2) Since $\bigwedge_{i\in I}\mu_i(x^{\Pi_{i\in I}a_i})=\inf\{\mu_i\}(x^{a_i})^{\Pi_{j\neq i}a_j}\,|\,j\in I\}=1$, $O_{\bigwedge_{i\in I}\mu_i}(x)$ is a divisor of $\Pi_{i\in I}a_i$ by Proposition 2.4.

Proposition 4.6. Let $\{\mu_i\,|\,\mu_i\in L^G, i\in I\}$ be a directed family of *TL*-p-subgroups of G with $T=\wedge$, where I is a nonempty index set. Then $\bigvee_{i\in I}\mu_i$ is a *TL*-p-subgroup.

Proof. Let $n=mp^t$, $(p, m)=1$. For each $i\in I$, since $(O_{\mu_i}(x^{p^t}), m)=1$ we have $\mu_i(x^{mp^t})=\mu_i(x^{p^t})$. Then $\bigvee_{i\in I}\mu_i(x^n)=\bigvee_{i\in I}\mu_i(x^{mp^t})=\bigvee_{i\in I}\mu_i(x^{p^t})$. Thus $\min\{n\,|\,\bigvee_{i\in I}\mu_i(x^n)=1\}$ is a power of p. Therefore, $O_{\bigvee_{i\in I}\mu_i}(x)$ is a power of p for all $x\in G$.

Proposition 4.7. Let $f: G\longrightarrow H$ be an epimorphism. And let $\mu_i\,|\,\mu_i\in L^G, i\in I\}$ be a directed family of L^H, where I is a nonempty index set. For each $i \in I$, let μ_i be f-invariant. Then $\{f(\mu_i) \mid \mu_i \in L^G, i \in I\}$ is a directed family of L^H.

Proof. Let $y \in H$. Then there exists an $x \in G$ such that $y = f(x)$. Since $\{\mu_i \mid \mu_i \in L^G, i \in I\}$ is directed, there exists an $i_x \in I$ such that $\mu_{i_x}(x) = \bigvee_{i \in I} \mu_i(x)$. Therefore there exists an $i_x \in I$ such that $f(\mu_{i_x})(y) = \bigvee_{i \in I} f(\mu_i)(y)$, since μ_i is f-invariant for each $i \in I$. For each pair $(i_1, i_2) \in I \times I$, since $\{\mu_i\} \mid \mu_i \in L^G, i \in I\}$ is directed, there exists an $i_3 \in I$ such that $\mu_{i_1} \vee \mu_{i_2} \leq \mu_{i_3}$. Therefore there exists an $i_3 \in I$ such that

$$f(\mu_{i_1}) \vee f(\mu_{i_2}) = f(\mu_{i_1} \vee \mu_{i_2}) \leq f(\mu_{i_3}).$$

This completes the proof.

Proposition 4.8. Let $f: G \to H$ be a homomorphism. And let $\{\mu_i\} \mid \mu_i \in L^H, i \in I\}$ be a directed family of L^H, where I is a nonempty index set. Then $f^{-1}(\mu_i) \mid \mu_i \in L^H, i \in I\}$ is a directed family of L^G.

Proof. Let $x \in G$. Since $\{\mu_i \mid \mu_i \in L^H, i \in I\}$ is directed, there exists an $i_{f(x)} \in I$ such that $\mu_{i_{f(x)}}(f(x)) = \bigvee_{i \in I} \mu_i(f(x))$. Therefore there exists an $i_{f(x)} \in I$ such that $f^{-1}(\mu_{i_{f(x)}})(x) = \bigvee_{i \in I} f^{-1}(\mu_i)(x)$. For each pair $(i_1, i_2) \in I \times I$, since $\{\mu_i\} \mid \mu_i \in L^H, i \in I\}$ is directed, there exists an $i_3 \in I$ such that $\mu_{i_1} \vee \mu_{i_2} \leq \mu_{i_3}$. Therefore there exists an $i_3 \in I$ such that

$$f^{-1}(\mu_{i_1}) \vee f^{-1}(\mu_{i_2}) = f^{-1}(\mu_{i_1} \vee \mu_{i_2}) \leq f^{-1}(\mu_{i_3}).$$

This completes the proof.

References

[1] Shih-Chuan Cheng, Zhudeng Wang, Divisible *TL*-subgroups and pure *TL*-subgroups, *Fuzzy Sets and Systems* **78**, 387-393 (1996).

[2] H.-D. Kim, Y.-H. Kim and J.-G. Kim, *TL*-subgroups having the property (∗), *Fuzzy Sets and Systems* (to appear).

[3] J.-G. Kim, Fuzzy subgroups and minimal fuzzy *p*-subgroups, *J. Fuzzy Math.* **2**, 913-921 (1994).

[4] J.-G. Kim, Orders of fuzzy subgroups and fuzzy *p*-subgroups, *Fuzzy Sets and Systems* **61**, 225-230 (1994).

[5] J.-G. Kim, Some properties of *TL*-groups, *Korean J. Comput. & Appl. Math.* **5**(2), 285-292 (1998).

[6] J.-G. Kim and H.-D. Kim, A characterization of fuzzy subgroups of some Abelian groups, Inform. Sci. **80**, 243-252 (1994).

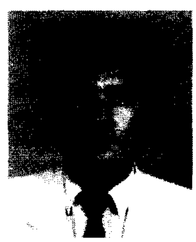
[7] J.-G. Kim and H.-D. Kim, Orders relative to *TL*-subgroups, *Math. Japon.* **46**, 163-168 (1997).

[8] J.-G. Kim and H.-D. Kim, Some characterizations of fuzzy subgroups: via fuzzy p^*-subsets and fuzzy p^*-subgroups, *Fuzzy Sets and Systems* (to appear).

[9] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35**, 512-517 (1971).

[10] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.*, **10**, 313-334 (1950).

[11] Y. Yu, J. N. Mordeson and S.-C. Cheng, Elements of *L*-algebra, Lecture Notes in Fuzzy Math. and Computer Science, Creighton Univ., Omaha, 1994.



김 한 두(Han-Doo Kim)
1982년 : 고려대학교 수학과(이학사)
1984년 : 고려대학교 수학과 대학원(이학석사)
1988년 : 고려대학교 수학과 대학원(이학박사)
1989년~현재 : 인제대학교 수학과 재직중(부교수)
주관심분야 : 퍼지이론, Automata 이론, 그래프론과 알고리즘

김 동 석(Dong-Seog Kim)
경성대학교 수학과 박사과정

김 재 겸(Jae-Gyeom Kim)
제8권 제4호 참조
현재 : 경성대학교 수학과 부교수