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A Note on Fuzzy *S*-mappings

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ABSTRACT

We introduce the concepts of fuzzy *s*-continuous mappings, *s*-open mappings, and *s*-closed mappings. We investigate several properties of such mappings. In particular, we study the relation between fuzzy *s*-continuous mappings and fuzzy *s*-open mappings (*s*-closed mappings).

1. Introduction

Fuzzy topological spaces were first introduced in the literature by Chang [1] who studied the basic concepts including fuzzy continuous mappings and compactness. And fuzzy topological spaces are a very natural generalization of topological spaces.

In 1983, A. S. Mashhour. et al. [3] introduced supratopological spaces and studied *s*-continuous mappings and *s**-continuous mappings. In 1987, M. E. Abd El-Monsef. et al. [2] introduced the fuzzy supratopological spaces and studied fuzzy supra-continuous functions and characterized the basic concepts. Also fuzzy supratopological spaces are a generalization of supratopological spaces. In this paper, we introduce the concepts of fuzzy *s*-continuous mappings, *s*-open mappings, and *s*-closed mappings. And we investigate several properties of such mappings. In particular, we introduce the relation between fuzzy *s*-continuous mappings and fuzzy *s*-open mappings (*s*-closed mappings).

Let X be a set and let $I=[0,1]$. Let I^X denote the set of all mappings $a: X \to I$. A number of I^X is called *a fuzzy set* of X. And unions and intersections of fuzzy sets are denoted by $\vee$ and $\wedge$ respectively and defined by

$$\vee a_i = sup\ \{a_i(x) \mid i \in J \text{ and } x \in X\},$$
$$\wedge a_i = inf\ \{a_i(x) \mid i \in J \text{ and } x \in X\}.$$

Definition 1.1. [1] *A fuzzy topology* T on X is a subfamily of I^X such that

(1) $0, 1 \in$ T

(2) if $a, b \in T$, then $a \wedge b \in T$

(3) if $a_i \in T$ for all $i \in J$, then $\vee a_i \in T$

The pair (X, T) is called *a fuzzy topological spaces* (*fts*, for short). Members of T are called *fuzzy open sets* in (X, T) and completement of a fuzzy open set is called *a fuzzy closed set.*

Definition 1.2. [5]. Let f be a mapping from a set X into a set Y. Let a and b be respectively the fuzzy sets of X and Y. Then $f(a)$ is a fuzzy set in Y, defined by

$$f(a)(y) = \begin{Bmatrix} \sup_{z \in f^{-1}(y)} a(z) & \text{if } f^{-1}(y) \neq \phi,\ y \in Y, \\ 0 & otherwise, \end{Bmatrix}$$

and $f^{-1}(b)$ is a fuzzy set in X, defined by $f^{-1}(b)(x)=b(f(x))$, for each $x \in X$.

Definition 1.3. [3]. A subfamily T^* of I^X is called *a fuzzy supratopology* on X if

(1) $1 \in T^*$

(2) if $a_i \in T^*$, $i \in J$, then, $\vee a_i \in T^*$.

The pair (X, T^*) is called *a fuzzy supratopological space*(*fsts*. for short) The elements of T^* are called *fuzzy supra-open sets* in (X, T^*). And a fuzzy set a is *supra-closed* if and only if $co(a)=1-a$ is a fuzzy supra-open set.

Definition 1.4. [3]. *The supra closure of a fuzzy set* a is denoted by $scl(a)$, and given by

$$scl(a) = \wedge\ \{ s \mid s \text{ is a fuzzy supra-closed set and } a \leq s\}.$$

The supra interior of a fuzzy set a is denoted by $si(a)$ and given by

$$si(a) = \wedge\ \{ t \mid t \text{ is a fuzzy supra-open set and } t \leq a\}.$$

2. Fuzzy s-continuous mappings

Definition 2.1. Let (X, T) be a fuzzy topological space and T^* be a fuzzy supratopology on X. We call T^* *an associated fuzzy supratopology with* T if $T \subseteq T^*$.

Definition 2.2. [2]. Let $f:(X, T^*) \to (Y, S^*)$ be a fuzzy mapping between two fuzzy supratopological spaces.

(1) f is called *a fuzzy supracontinuous mapping if* $f^{-1}(S^*) \subseteq T^*$

(2) f is called *a fuzzy supraopen mapping* if $f(T^*) \subseteq S^*$.

Definition 2.3. Let (X, T) and (Y, S) be fuzzy topological spaces and T^* be an associated fuzzy supratopology with T. A mapping $f: X \to Y$ is said to be *fuzzy s-continuous* if for each fuzzy open set a in Y, $f^{-1}(a)$ is a fuzzy supra-open set in (X, T^*).

We recall that a fuzzy set h in a fts (X, T) is a neighborhood of a fuzzy set f in X if and only if there is $g \in T$ such that $f \leq g \leq h$.

Theorem 2.4. Let (X, T) and (Y, S) be fuzzy topological spaces and let T^* be an associated fuzzy supratopology with T. If f is a mapping from X into Y, then the followings are equivalent :

(1) f is fuzzy s-continuous.

(2) For each fuzzy closed set a in Y, $f^{-1}(a)$ is a fuzzy supra-closed set in (X, T^*).

(3) $scl(f^{-1}(a)) \leq f^{-1}(cl(a))$ for every fuzzy set a in Y.

(4) $f(scl(a)) \leq cl(f(a))$ for every fuzzy set a in X.

(5) $f^{-1}(int(b)) \leq si(f^{-1}(b))$ for every fuzzy set b in Y.

(6) For each fuzzy set a in X and each neighborhood b of $f(a)$, there is a supra neighborhood c of a such that $f(c) \leq b$.

Proof. (1) $\Rightarrow$ (2). Let a be fuzzy closed set in Y. Since f is fuzzy s-continuous, $f^{-1}(1-a)=1-f^{-1}(a)$ is a fuzzy supra-open in X. Therefore $f^{-1}(a)$ is a fuzzy supra-closed set in X.

(2) $\Rightarrow$ (3). Since $cl(a)$ is fuzzy closed for every fuzzy set a in Y, $f^{-1}(cl(a))$ is a fuzzy supra-closed set. Therefore,

$$f^{-1}(cl(a))=scl(f^{-1}cl(a)) \geq scl(f^{-1}(a)).$$

(3) $\Rightarrow$ (4). Let a be a fuzzy set in X and let $f(a)=b$. Then $f^{-1}(cl(b)) \geq scl(f^{-1}(b))$. So $f^{-1}(cl(a)) \geq scl(f^{-1}f(a)) \geq scl(a)$, and hence $cl(f(a)) \geq f(scl(a))$.

(4) $\Rightarrow$ (2). Let b be a fuzzy closed set in Y and $a=f^{-1}(b)$. Then $f(scl(a)) \leq cl(f(a))=cl(f(f^{-1}(b))) \leq cl(b)=b$. Since $scl(a) \leq f^{-1}(f(scl(a))) \leq f^{-1}(b)=a$, a is a fuzzy supra-closed.

(2) $\Rightarrow$ (1). It is obvious.

(1) $\Rightarrow$ (5). Let b be a fuzzy subset in Y. Since $f^{-1}(int(b))$ is a fuzzy supra-open set in X, $f^{-1}(int(b)) \leq si(f^{-1}(int(b)) \leq si(f^{-1}(b))$.

(5) $\Rightarrow$ (1). Let a be a fuzzy open in Y. Since $f^{-1}(a) \leq si(f^{-1}(a)) \leq f^{-1}(a)$, $f^{-1}(a)$ is a fuzzy supra-open set.

(6) $\Rightarrow$ (1). Let b be any fuzzy open set in Y and let $f^{-1}(b)=a$. Then b is a neighborhood of $f(a)=f(f^{-1}(b))$. There exists a supra neighborhood c of $a=f^{-1}(b)$ such that $f(c) \leq b$. Thus $c \leq f^{-1}f(c) \leq f^{-1}(b)$. Therefore, $f^{-1}(b)$ is a supra neighborhood of $f^{-1}(b)$. And $f^{-1}(b)$ is a fuzzy supra-open set in X by [3, Theorem 2.2].

(1) $\Rightarrow$ (6). It is obvious.

Remark. Every fuzzy continuous mapping is fuzzy s-continuous. But the converse of this implication is not true, as following example shows.

Example 2.1. Let a_1, a_2, and a_3 be fuzzy sets of $X = I$ defined as

$$a_1(x)=\begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x-1, & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$a_2(x)=\begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x+2, & \text{if } \frac{1}{2} \leq x \leq 1; \\ 0, & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$a_3(x)=\begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ -2x+2, & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

Consider the fuzzy topology $T=\{0, a_1, a_2, a_1 \vee a_2, 1\}$ and an associated fuzzy supratopology $T^*=\{0, a_1, a_2, a_3, a_1 \vee a_2, a_1 \vee a_3, 1\}$. Let the mapping $g:X \to X$ be defined by $g(x)=(1/2)x$. Clearly, we have $g^{-1}(0)=0$, $g^{-1}(1)=1$, $g^{-1}(a_1 \vee a_2)=a_3$, $g^{-1}(a_2)=a_3$ and $g^{-1}(a_1)=0$. The fuzzy set a_3 is fuzzy supra-open in (X, T^*) but it is not fuzzy open in (X, T). Hence the mapping g is fuzzy s-cintinuous but not fuzzy continuous.

Remark. In general, the composition of two fuzzy s-continuous mappings need not be fuzzy s-continuous.

Example 2.2. Let $X=I$. Consider the fuzzy sets

$$a(x)=\begin{cases} 1, & \text{if } 0 \le x < \frac{1}{3}, \\ \frac{1}{2}, & \text{if } \frac{1}{3} \le x < \frac{2}{3}, \\ 0, & \text{if } \frac{2}{3} \le x < 1; \end{cases}$$

$$b(x)=\frac{1}{2}, \quad \text{if } 0 \le x \le 1;$$

$$c(x)=\frac{1}{3}, \quad \text{if } 0 \le x \le 1.$$

Let $T_1=\{0, a, 1\}$ and $T_1^*=\{0, a, b, a \vee b, 1\}$. Let $T_2=\{0, c, 1\}$ and $T_2^*=\{0, a, c, a \vee c, 1\}$. Let $f:(X, T_1) \to (X, T_1)$ be the mapping defined by $f(x)=(x+1)/3$. Let $g:(X, T_2) \to (X, T_1)$ be the mapping defined by $g(x)=(1/3)x$. Clearly, f and g are fuzzy s-continuous. But $f \circ g$ is not fuzzy s-continuous, since a is a fuzzy open set in (X, T_1) but $(f \circ g)^{-1}(a)=b$ is not fuzzy supra-open in T_2^*.

Theorem 2.5. If a mapping $f:(X, T_1) \to (Y, T_2)$ is fuzzy s-continuous and a mapping $g:(Y, T_2) \to (Z, T_3)$ is fuzzy continuous, then $g \circ f$ is fuzzy s-continuous.

Proof. It is clear by the definitions of fuzzy s-continuous mappings and fuzzy continuous mappings.

Theorem 2.6. Let (X, T) and (Y, S) be two fuzzy topological spaces, T^* and S^* be two associated fuzzy supratopologies with T and S, respectively. A mapping $f:X \to Y$ is fuzzy continuous if it has one of the following properties:

(1) $f^{-1}(si(a)) \le int(f^{-1}(a))$ for each fuzzy set a in (Y, S).

(2) $cl(f^{-1}(a)) \le f^{-1}(scl(a))$ for each fuzzy set a in (Y, S).

(3) $f(cl(b)) \le scl(f(b))$ for each fuzzy set b in (X, T).

Proof. If the condition (2) is satisfied, let b be a fuzzy closed set in Y, then $cl(f^{-1}(b)) \le f^{-1}(scl(b))=f^{-1}(b)$. Therefore $f^{-1}(b)$ is a fuzzy closed set in X. If the condition (3) is satisfied, let b be a fuzzy set in Y, then $f^{-1}(b)$ is a fuzzy set in X and $f(cl(f^{-1}(b))) \le scl(f(f^{-1}(b)))$. Thus $cl(f^{-1}(b)) \le f^{-1}(scl(b))$.
Therefore, since the condition (2) in satisfied, f is continuous.
In case (1), we can prove similarly.

Lemma [4]. Let $g:X \to X \times Y$ be the graph of a mapping $f:X \to Y$. Then, if a is a fuzzy set in X and b is a fuzzy set in Y, $g^{-1}(a \times b)=a \wedge f^{-1}(b)$.

Theorem 2.7. Let $f:(X, T) \to (Y, S)$ be a mapping and T^* be an associated supratopology with T. Let $g: X \times Y$, given by $g(x)=(x, f(x))$ be its graph mapping. If g is fuzzy s-continuous, then f is fuzzy s-continuous.

Proof. Suppose that g is a fuzzy s-continuous and a is a fuzzy open set in (Y, S). Then $f^{-1}(a)=1 \wedge f^{-1}(a)$: $g^{-1}(1 \times a)$. Therefore, $f^{-1}(a)$ is a fuzzy supra-open set in (X, T^*).

Definition 2.8. [3]. Let (X, T_1) and (Y, T_2) be fuzzy topological spaces and T_1^* and T_2^* be two associated fuzzy supratopologies with T_1 and T_2, respectively. Let $f:(X, T_1) \to (Y, T_2)$ be a mapping. Then f is said to be ***fuzzy supracontinuous*** if for each fuzzy supra-open set a in (Y, T_2^*), $f^{-1}(a)$ is a fuzzy supra-open set in (X, T_1^*).

Remark. Every fuzzy supracontinuous mappings is fuzzy s-continuous, but the converse is not true. Let $f: X \to Y$ and $g: Y \to Z$ be two mappings. If f is a fuzzy supra-continuous and g is fuzzy s-continuous, then $g \circ f: X \to Z$ is fuzzy s-continuous.

3. Fuzzy s-open mappings and Fuzzy s-closed mappings.

Definition 3.1. A mapping $f:(X, T)\to(Y, S)$ is said to be *fuzzy s-open* (*fuzzy s-closed*, respectively) if the image of each fuzzy open (fuzzy closed, repectively) set in (X, T) is fuzzy supra-open (fuzzy supra-closed, respectively) in (Y, S^*).

Now we recall that a mapping $f:(X, T)\to(Y, S)$ is said to be *fuzzy open* if $f(T)\subseteq S$. Clearly, every fuzzy open (fuzzy closed) mapping is a fuzzy s-open mapping (fuzzy s-closed mapping). And every fuzzy supraopen mapping is a fuzzy s-open mapping. But the converses of these implications are not true, which are from the following examples.

Example. Let $X = I$. Consider the fuzzy sets ;

$$a(x)=\begin{cases} 2x, & \text{if } 0\le x\le \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2}<x\le 1; \end{cases}$$

$$b(x)=\begin{cases} \frac{1}{2}, & \text{if } 0\le x<\frac{1}{4}, \\ 2x, & \text{if } \frac{1}{4}\le x\le \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2}<x\le 1; \end{cases}$$

$$c(x)=\begin{cases} 1, & \text{if } 0\le x\le \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2}<x\le 1. \end{cases}$$

(1). Let $T=\{0, a, 1\}$ be a fuzzy topology on X and $T^* = \{0, a, b, c, a\vee c, a\vee b, 1\}$ be an associated fuzzy supratopology with T. Let $f:(X, T)\to(X, T)$ be the mapping defined by

$$f(x)=\begin{cases} x, & \text{if } 0\le x\le \frac{1}{2}, \\ 1-x, & \text{if } \frac{1}{2}<x\le 1. \end{cases}$$

Now we show that $f(a)=b$.

Case 1. Let $0\le x<\frac{1}{4}$ and $y\in f^{-1}(x)$. Then either $0\le y<\frac{1}{4}$ or $\frac{3}{4}<y\le 1$. Since

$$a(y)=\begin{cases} 2y, & \text{if } 0\le y<\frac{1}{4}, \\ \frac{1}{2}, & \text{if } \frac{3}{4}<y\le 1, \end{cases}$$

and $2y\le\frac{1}{2}$ for $y\in f^{-1}(x)$. Therefore $f(a)(y)=\frac{1}{2}$ if $0\le x<\frac{1}{4}$.

Case 2. Let $\frac{1}{4}\le x\le\frac{1}{2}$ and y in $f^{-1}(x)$. Then either $\frac{1}{4}\le y\le\frac{1}{2}$ or $\frac{1}{2}<y\le\frac{3}{4}$. Thus

$$a(y)=\begin{cases} 2y, & \text{if } \frac{1}{4}\le y\le \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2}<y\le \frac{3}{4}. \end{cases}$$

Since $\frac{1}{2}\le 2y$ and $y=x$ for each $\frac{1}{4}\le y\le\frac{1}{2}$, we obtain $f(a)(y)=2x$ if $\frac{1}{4}\le x\le\frac{1}{2}$.

Case 3. Let $\frac{1}{2}<x\le 1$. Since $f^{-1}(x)$ is empty set for $\frac{1}{2}<x\le 1$, $f(a)=0$ if $\frac{1}{2}<x\le 1$.

Therefore by the above facts, we obtain $f(a)=b$.
And similarly, we have $f(1)=c$. Since b and c are two fuzzy supra-open sets in T^*, f is a fuzzy s-open mapping. But since b and c are not fuzzy open in T, f is not a fuzzy open mapping.

(2). Let $T=\{0, b, 1\}$ be a fuzzy topology on X. Let $T^*=\{0, a, b, a\vee b, 1\}$ and $S^*=\{0, b, c, 1\}$ are associated fuzzy supratopologies with T. Consider the fuzzy mapping $f:(X, T^*)\to(X, S^*)$ defined by

$$f(x)=\begin{cases} x, & \text{if } 0\le x\frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2}<x\le 1. \end{cases}$$

We obtain $f(b)=b$ and $f(1)=c$. Thus f is a fuzzy s-open mapping. But for a fuzzy supra-open set a in T^*, $f(a)$ is not fuzzy supra-open in S^*. Consequently, f is not a fuzzy supraopen mapping.

Theorem 3.2. Let $f:(X, T_1)\rightarrow(Y, T_2)$ be a mapping. Then the followings are equivalent:

(1) f is a fuzzy s-open mapping.

(2) $f(int(a)) \leq si\,(f(a))$ for each fuzzy set a in X.

Proof. (1) $\Rightarrow$ (2). Since $\mathrm{int}(a) \leq a$, we have $f(int(a)) \leq f(a)$. By hypothesis, $f(int(a))$ is supra-open in $f(a)$. And since $si(f(a))$ is the largest fuzzy supra-open set in $f(a)$, we obtain $f(int(a)) \leq si(f(a))$.

(2) $\Rightarrow$ (1). Let a be a fuzzy open in X. We have $si\,(f(a)) \leq f(a)$. By the hypothesis, $f(a) \leq si(f(a))$. Thus $f(a)$ is a fuzzy supra-open set in Y.

Theorem 3.3. Let $f:(X, T_1)\rightarrow(Y, T_2)$ be a mapping. Then f is fuzzy s-closed if and only if $scl(f(a)) \leq f(cl(a))$ for each a in X.

Proof. If f is fuzzy s-closed mapping, then $f(cl(a))$ is a fuzzy supra-closed set in Y. And we have $f(a) \leq f(cl(a))$, thus $scl(f(a)) \leq f(cl(a))$.
Conversely, let a be a fuzzy closed set. Then $f(a) \leq scl(f(a)) \leq f(cl(a)) = f(a)$, thus $f(a)$ is a fuzzy supra-closed set in Y.

Theorem 3.4. Let $f:(X, T_1)\rightarrow(Y, T_2)$ and $g:(Y, T_2)\rightarrow(Z, T_3)$ be two mappings.

(1) If $g \circ f$ is fuzzy s-open and f is fuzzy continuous surjective, then g is also fuzzy s-open.

(2) If $g \circ f$ is fuzzy open mapping and g is fuzzy s-continuous injective, then f is fuzzy s-open.

Proof. (1). Let a be a fuzzy open set in Y. Then $f^{-1}(a)$ is fuzzy open in X. Since $(g \circ f)$ is fuzzy s-open, $(g \circ f)(f^{-1}(a))$ is a supra-open set in Z. And $(g \circ f)(a \circ f) = g(a)$, since f is surjective. Therefore the mapping g is fuzzy s-open.
(2). Let a be a fuzzy open set in X. Then $g \circ f(a) = g(f(a))$ is fuzzy open in Z. Since g is fuzzy s-continuous injective, $g^{-1}(g(f(a))) = g(f(a)) \circ g = f(a)$ is a fuzzy supra-open set. Hence, f is fuzzy s-open.

Theorem 3.5. Let (X, T_1) and (Y, T_2) be two fuzzy topological spaces. If $f:(X, T_1)\rightarrow(Y, T_2)$ is a bijective mapping, then following statements are equivalent :

(1) f is a fuzzy s-open mapping.

(2) f is a fuzzy s-closed mapping.

(3) f^{-1} is fuzzy s-continuous.

Proof. (1) $\Rightarrow$ (2). Let a be a fuzzy closed set in X. Then $f(1\text{-}a) = 1\text{-}f(a)$ is fuzzy supra-open in Y, since f is a fuzzy s-open mapping. Hence $f(a)$ is fuzzy supra-closed in Y.

(2) $\Rightarrow$ (3). Let a be a fuzzy closed set in X. We have $(f^{-1})^{-1}(a) = f(a)$. Since f is fuzzy s-closed mapping, $f(a)$ is fuzzy supra-closed in Y. Therefore, f is fuzzy s-continuous.

(3) $\Rightarrow$ (1). Let a be a fuzzy open set in X. Since f^{-1} is fuzzy s-continuous, $(f^{-1})^{-1}(a) = f(a)$ is fuzzy supra-open in Y. Hence f is a fuzzy s-open mapping.

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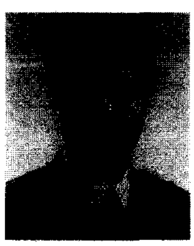

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