THE STABILITY OF THE EQUATION \( f(x + p) = kf(x) \)

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ABSTRACT. In this paper, we investigate the Hyers-Ulam stability of the \((p, k)\)-MP functional equation.

1. Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam [5]. In 1940, he had raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping? In 1941, this problem was solved by D. H. Hyers [1] for the first time. This problem has been further generalized and solved by Th. M. Rassias [4]. Thereafter, the stability problem of functional equations has been extended in various directions and studied by several mathematicians [2].

In this paper, the Hyers-Ulam stability of the \((p, k)\)-MP functional equation (1) is investigated. Furthermore, a modified Hyers-Ulam-Rassias stability of the functional equation (9) shall also be investigated.

2. Hyers-Ulam stability of the \((p, k)\)-MP functional equation

The following functional equation

\[
(1) \quad f(x + p) = kf(x)
\]

is called the \((p, k)\)-MP functional equation. Throughout this section, let \(\delta > 0\), \(k > 0\), and \(p \neq 0\) be fixed.

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Theorem 1. Let $k \neq 1$. If a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality

\begin{equation}
|f(x + p) - kf(x)| \leq \delta
\end{equation}

for all $x \in \mathbb{R}$, then there exists a unique solution $F : \mathbb{R} \to \mathbb{R}$ of the $(p, k)$-MP functional equation (1) with

\begin{equation}
|F(x) - f(x)| \leq |k - 1|^{-1}\delta
\end{equation}

for all $x \in \mathbb{R}$.

Proof. (1) The case of $1 < k$: For any $x \in \mathbb{R}$ and for every non-negative integer $n$ we define

$$P_n(x) = k^{-n}f(x + pn).$$

Then $P_0(x) = f(x)$. By (2) we have

$$|P_{n+1}(x) - P_n(x)| \leq k^{-(n+1)}\delta,$$

whence

\begin{equation}
|P_n(x) - f(x)| \leq (k - 1)^{-1}\delta.
\end{equation}

Let $m \leq n$. Then

$$|P_n(x) - P_m(x)| \leq k^{-m}(k - 1)^{-1}\delta,$$

whence $|P_n(x) - P_m(x)| \to 0$ as $m, n \to \infty$ since $k > 1$. This fact implies that $\{P_n(x)\}$ is a Cauchy sequence for all $x \in \mathbb{R}$ and hence we can define a mapping $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \lim_{n \to \infty} P_n(x)$$

for all $x \in \mathbb{R}$. Then by (4), $F$ satisfies the inequality (3) and

\begin{equation}
F(x + p) = \lim_{n \to \infty} P_{n-1}(x + p) = k \lim_{n \to \infty} P_n(x) = kF(x)
\end{equation}
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for all $x \in \mathbb{R}$. Thus $F(x + pn) = k^n F(x)$, so $F(x) = k^{-n} F(x + pn)$. Now, let $G : \mathbb{R} \to \mathbb{R}$ be another mapping which satisfies (5) as well as (3) for all $x \in \mathbb{R}$. It follows from (5) and (3) that

$$|F(x) - G(x)| = |k^{-n} F(x + pn) - k^{-n} G(x + pn)|$$

$$\leq k^{-n}(|F(x + pn) - f(x + pn)|$$

$$+ |G(x + pn) - f(x + pn)|)$$

$$\leq k^{-n}((k - 1)^{-1} \delta + (k - 1)^{-1} \delta)$$

for all $x \in \mathbb{R}$ and all positive integers $n$. Thus $F(x) = G(x)$.

(II) The case of $0 < k < 1$: An equivalent formula of inequality (2) is

$$(6) \quad |f(x - p) - k^{-1} f(x)| \leq k^{-1} \delta.$$ 

By the proof of the case (I), there exists unique $F : \mathbb{R} \to \mathbb{R}$ such that

$$(7) \quad F(x - p) = k^{-1} F(x)$$

and

$$(8) \quad |F(x) - f(x)| \leq \frac{1}{k^{-1} - 1} k^{-1} \delta = \frac{1}{1 - k} \delta.$$

An equivalent form of (7) is

$$F(x + p) = kF(x).$$

The proof is complete. \qed

The following example shows that the above theorem is false when $k = 1$.

**Example 2.** Let $f(x) = x$ and $p = 1$, $\delta = 1$, $k = 1$. Then $|f(x + 1) - f(x)| = 1 = \delta$. Assume that $F(x)$ is a solution of the $(1, 1)$-MP functional equation (1). If $F(0) = c$, then $F(n) = c$ for all $n \in \mathbb{N}$. Thus $|F(n) - f(n)| = |c - n| \to \infty$ as $n \to \infty$. 

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3. A modified Hyers-Ulam-Rassias stability of the \((p, k)\)-MP functional equation

Let \(\delta, \epsilon, p > 0\) be given and define
\[
\alpha(x) = \prod_{i=0}^{\infty} [1 - \delta(x + pi)^{-(1+\epsilon)}], \quad \beta(x) = \prod_{i=0}^{\infty} [1 + \delta(x + pi)^{-(1+\epsilon)}]
\]
for any \(x > \delta^{1/(1+\epsilon)}\). Let \(n_0 \geq 0\) be any integer. By using an idea from [3], we can prove the following theorem:

**Theorem 3.** Let \(0 < k\). If a mapping \(f : (0, \infty) \to (0, \infty)\) satisfies the inequality
\[
\left| \frac{f(x + p)}{kf(x)} - 1 \right| \leq \frac{\delta}{x^{1+\epsilon}}
\]
for all \(x > n_0\), then there exists a unique solution \(F : (0, \infty) \to [0, \infty)\) of the \((p, k)\)-MP functional equation (1) with
\[
\alpha(x) \leq F(x)/f(x) \leq \beta(x)
\]
for all \(x > \max\{n_0, \delta^{1/(1+\epsilon)}\}\).

**Proof.** Let \(P_n(x)\) be defined as in the proof of Theorem 1. For any \(x > 0\) and for all positive integers \(m, n\) with \(n > m\), it holds
\[
\frac{P_n(x)}{P_m(x)} = \frac{f(x + p(m + 1))}{kf(x + pm)} \frac{f(x + p(m + 2))}{kf(x + p(m + 1))} \cdots \frac{f(x + pn)}{kf(x + p(n - 1))}.
\]
If \(m > n_0\) is so large that \(1 - \delta(x + pm)^{-(1+\epsilon)} > 0\), we then obtain
\[
\prod_{i=m}^{n-1} [1 - \delta(x + pi)^{-(1+\epsilon)}] \leq P_n(x)/P_m(x) \leq \prod_{i=m}^{n-1} [1 + \delta(x + pi)^{-(1+\epsilon)}]
\]
or
\[
\sum_{i=m}^{n-1} \ln [1 - \delta(x + pi)^{-(1+\epsilon)}] \leq \ln P_n(x) - \ln P_m(x) \leq \sum_{i=m}^{n-1} \ln [1 + \delta(x + pi)^{-(1+\epsilon)}].
\]
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Since

$$\lim_{m \to \infty} \sum_{i=m}^{\infty} |\ln [1 - \delta(x+pi)^{-(1+\epsilon)}]| = \lim_{m \to \infty} \sum_{i=m}^{\infty} \ln [1 + \delta(x+pi)^{-(1+\epsilon)}] = 0,$$

we conclude that $\{\ln P_n(x)\}$ is a Cauchy sequence for all $x > 0$. Hence, we can define

$$L(x) = \lim_{n \to \infty} \ln P_n(x)$$

and

$$F(x) = e^{L(x)}$$

for all $x > 0$. So, $F(x) = \lim_{n \to \infty} P_n(x)$ and

$$F(x + p) = \lim_{n \to \infty} P_n(x + p) = \lim_{n \to \infty} kP_{n+1}(x) = kF(x)$$

for all $x > 0$. Now, let $x > \max\{n_0, \delta^{1/(1+\epsilon)}\}$. It then holds $1 - \delta(x + pi)^{-(1+\epsilon)} > 0$ for $i = 0, 1, \cdots$. Therefore, it follows from (9) that

$$\prod_{i=0}^{n-1} [1 - \delta(x + pi)^{-(1+\epsilon)}] \leq P_n(x)/f(x) \leq \prod_{i=0}^{n-1} [1 + \delta(x + pi)^{-(1+\epsilon)}]$$

since

$$\frac{P_n(x)}{f(x)} = \frac{f(x + pn)}{kf(x + p(n-1))} \frac{f(x + p(n-1))}{kf(x + p(n-2))} \cdots \frac{f(x + p)}{kf(x)}.$$ 

This implies the validity of (10). Now, it remains only to prove the uniqueness of $F$. Assume that $G : (0, \infty) \to [0, \infty)$ is another solution of the $(p, k)$-MP functional equation (1) and satisfies (10). Since both $F$ and $G$ are solutions of (1), it follows

$$\frac{F(x)}{G(x)} = \frac{F(x + pn)}{G(x + pn)} = \frac{F(x + pn)}{f(x + pn)} \frac{f(x + pn)}{G(x + pn)}$$

for any $x > 0$. Hence, we have

$$\frac{\alpha(x + pn)}{\beta(x + pn)} \leq \frac{F(x)}{G(x)} \leq \frac{\beta(x + pn)}{\alpha(x + pn)}$$
for all sufficiently large $n$. It is clear that the infinite products $\alpha(x)$ and $\beta(x)$ converges for all $x > 0$. Therefore, by using the relations

$$\alpha(x) = \lim_{n \to \infty} \alpha(x+pn) \lim_{n \to \infty} \prod_{i=0}^{n-1} [1-\delta(x+pi)^{-(1+\epsilon)}] = \lim_{n \to \infty} \alpha(x+pn)\alpha(x)$$

and

$$\beta(x) = \lim_{n \to \infty} \beta(x+pn) \lim_{n \to \infty} \prod_{i=0}^{n-1} [1+\delta(x+pi)^{-(1+\epsilon)}] = \lim_{n \to \infty} \beta(x+pn)\beta(x),$$

we conclude that $\alpha(x+pn) \to 1$ and $\beta(x+pn) \to 1$ as $n \to \infty$. Hence, it is obvious that $F(x) = G(x)$ holds true for all $x > 0$. □

References


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