BANACH SUBSPACES AND ENVELOPE NORM OF $wL_1$

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ABSTRACT. In this paper as a universal Banach space of the separable Banach spaces we investigate the complemented Banach subspaces of $wL_1$. Also, using Peck's theorem and the properties of the envelope norm of $wL_1$ we will find a canonical basis of $l_1^n, l_\infty^n$ for each $n$.

1. Introduction

The space $weakL_1$ was introduced in analysis because key operators of harmonic analysis do not map $L_1$ into $L_1$. Examples of such operators are the Hardy-Littlewood maximal function and the Hilbert transform. In this viewpoint, it became natural to investigate $weakL_1$, the space of measurable functions $f$ satisfying $\mu(\{x \in \Omega : |f(x)| > y\}) \leq \frac{c}{y}$.

It is known that (except for some trivial measure space), $weakL_1$ is not normable (see [3]). The question therefore arise as to whether any nontrivial continuous linear functionals on $weakL_1$ exists. Also, one can ask the structures of $weakL_1$. In [3], the answer for this question was considered. This implies $weakL_1$ has nontrivial dual space. In [4], J. Kupka and T. Peck studied the structure of $weakL_1$ and showed that the space $L_\infty$ is dense in the dual of $weakL_1$ endowed with $weak^*$-topology.

As a Lorentz space, we will study the space $L(1, \infty)$ which is called $weakL_1$ denoted by $wL_1$:

\begin{equation}
(1.1) \quad wL_1 = \{ f \in L_0 : \mu(\{x \in \Omega : |f(x)| > y\}) < \frac{c}{y} \},
\end{equation}

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where \( c > 0 \) is independent of \( y > 0 \). As we mentioned in the above, \( wL_1 \) is not normable, but we can find nontrivial linear functionals on \( wL_1 \). This was first observed by M. Cwikel and Y. Sagher in [3].

In [1], if \( \mu \) is nonatomic, then we can get an equivalent integral-like seminorm
\[
(1.2) \quad \|f\|_{wL_1} = \lim_{n \to \infty} \sup_{\frac{1}{p} \geq \frac{n}{q}} \frac{1}{\log \frac{q}{p}} \int_{\{p \leq |f| \leq q\}} |f| d\mu.
\]

Later on, in [2] actually the Banach envelope seminorm on \( wL_1 \) was calculated to be exactly as above. Note that the seminorm on \( wL_1 \) defined in (1.2) is a lattice seminorm. Even though \( wL_1 \) is complete with respect to the quasinorm \( q_1(f) = \sup_{a > 0} \mu(\{x \in \Omega : |f(x)| > a\}) \), it is not complete with respect to the seminorm \( \| \cdot \|_{wL_1} \). This is due to M. Cwikel and C. Feffreman([1] and [4]). Let \( \mathcal{N} = \{f \in wL_1 : \|f\|_{wL_1} = 0\} \). Then we obtain the quotient space \( wL_1/\mathcal{N} \). We define \( wL_1 \) as the normed envelope (and its completion as the Banach envelope) of \( wL_1 \).

To study this subject, we need some basic facts about the dual of \( wL_1 \). We would like to convert the nonlinear limit superior expression (1.2) for \( \| \cdot \|_{wL_1} \) into a linear limit expression by assigning the numbers
\[
I^b_a(f) = \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} |f| d\mu
\]
in some fashion. For this, we introduce an ultrafilter \( \mathcal{U} \) so that the limit of the \( I^b_a \) along \( \mathcal{U} \) determines a canonical integral-like linear functional \( I_\mathcal{U} \in wL_1^* \). We now begin with the discussion of \( \mathcal{U} \). For \( n = 1, 2, 3, \ldots \), let
\[
(1.3) \quad F_n = \{(a, b) : 1 \leq a < b, \frac{b}{a} \geq n\}.
\]
and then define \( \mathcal{F} = \{F_n : n \geq 1\} \). Treating \( \mathcal{F} \) as a filter of subsets of the set \( S = [1, \infty) \times [1, \infty) \), we obtain from Zorn's lemma an ultrafilter \( \mathcal{U} \) of subsets of \( S \) such that \( \mathcal{F} \subset \mathcal{U} \).

From now on, we will fix the ultrafilter \( \mathcal{F} \subset \mathcal{U} \). Define the "ersatz integral" \( I_\mathcal{U} \) for every nonnegative function \( f \in wL_1 \) by
\[
(1.4) \quad I_\mathcal{U}(f) = \lim_{\mathcal{U}} I^b_a(f) = \lim_{\mathcal{U}} \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} f d\mu.
\]

J. Kupka and T. Peck gave the properties of ersatz integral \( I_\mathcal{U} \) as following ; see [4].
**THEOREM 1.1 (J. KUPKA AND T. PECK).** Let \( f, g \in wL_1 \) be non-negative and let \( r > 0 \). Then we have

i) \( I_\mu(rf) = rI_\mu(f) \),

ii) \( I_\mu(f + g) = I_\mu(f) + I_\mu(g) \),

iii) If \( f \leq g \), then \( I_\mu(f) \leq I_\mu(g) \),

iv) \( I_\mu(f) \leq \|f\|_{wL_1} \).

From these properties, we define \( I_\mu(f) \) for an arbitrary function \( f \in wL_1 \) by

\[
I_\mu(f) = \lim_{\mu} \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} (f^+ - f^-)d\mu;
\]

i) \( I_\mu \) is linear.

ii) \( |I_\mu(f)| \leq \|f\|_{wL_1} \) for all \( f \in wL_1 \).

iii) \( I_\mu \) vanishes on \( N = \{ f \in wL_1 : \|f\|_{wL_1} = 0 \} \) and hence determines a well defined, bounded linear functional on \( wL_1 \).

Define \( wL_1(\mu) = \{ f \in wL_1 : \|f\|_\mu < \infty \} \) where \( \mu \) is the ultrafilter defined in (1.3). Then we have \( \|f\|_\mu \leq \|f\|_{wL_1} \) where

\[
(1.5) \quad \|f\|_\mu = I_\mu(|f|) = \lim_{\mu} \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} |f|d\mu.
\]

Hence we have \( wL_1 \subset wL_1(\mu) \). Moreover \( \|f\|_\mu = I_\mu(f) \) has the following properties:

i) \( \| \cdot \|_\mu \) is a lattice seminorm on \( wL_1 \),

ii) \( \|f + g\|_\mu = \|f\|_\mu + \|g\|_\mu \) whenever \( f \) and \( g \) are nonnegative,

iii) \( \|f\|_{wL_1} = \sup\{\|f\|_\mu : \mu \text{ is an ultrafilter, } \mathcal{F} \subset \mu\} \) for all \( f \in wL_1 \).

Again, we convert \( \| \cdot \|_\mu \) into a norm by forming the ideal

\[
(1.6) \quad N_\mu = \{ f \in wL_1 : \|f\|_\mu = 0 \}.
\]

and then the quotient vector lattice \( wL_1(\mu) = wL_1/N(\mu) \) on which \( \| \cdot \|_\mu \) acts as a lattice norm.

We now have an information on the dual of \( wL_1 \):
Theorem 1.2 (J. Kupka and T. Peck). Define a linear operator $T_U : L_\infty \rightarrow wL_1^*$ by $T_U(m) : f \mapsto I_U(mf)$ for all $m \in L_\infty(\mu)$, and for all $f \in wL_1$. Then $T_U$ constitutes an isometric, order isomorphism of $L_\infty(\mu)$ into $wL_1^*$. Moreover, the linear span of the subspace $T_U(L_\infty(\mu))$, as $U$ ranges over the collection of ultrafilters which contains $F$, constitutes a norming and hence a weak* dense subspace of $wL_1^*$.

This theorem gives some favorable information for $wL_1(U)^*$ and the very last part of theorem says

$$(1.7) \quad \overline{T_U(B_{L_\infty})wL_1(U)^*} = B_{wL_1(U)^*}. $$

From this theorem, for any $m \in L_\infty(\mu)$, we have

$$(1.8) \quad \hat{m} = T_U(m) \in wL_1(U)^*. $$

Clearly, every linear functional $\varphi \in wL_1(U)^*$ is a linear functional on $wL_1$ (see more detail in [4, 2.20]).

We now give a lemma about linear functionals on $wL_1$ which is actually due to J. Kupka and T. Peck (see [4, 2.20]).

Lemma 1.3. For an ultrafilter $U$ containing $F$ in (1.3), let $f \in wL_1$ be a nonnegative function with $\|f\|_U = 1$. Then for any $g \in wL_1$, disjointly supported from $f$, there exists a $\phi \in wL_1^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$, and $\phi(g) = 0$.

We can now generalize this lemma for arbitrary pairwise disjointly supported elements in $wL_1$.

Corollary 1.4. Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative elements in $wL_1$ with $\|f_n\|_{wL_1} = 1$, for all $n = 1, 2, 3, \ldots$ and such that the $f_n$ have pairwise disjoint supports. Then for each $n$, there exists a linear functional $\phi_n$ on $wL_1$ such that $\phi_n(f) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$ if $n \neq m$. 

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**Proof.** We can show this by induction with lemma 1.3 for each \( f_n \). For given \( f_1 \in wL_1 \), by lemma 1.3, we can choose \( \phi_1 \) with \( \phi_1(f_1) = \|\phi\| = 1 \) and \( \phi_1(f_j) = 0 \), for all \( j = 2, 3, \ldots \). If we selected \( \phi_1, \phi_2, \ldots, \phi_n \) satisfying all the conclusions of corollary, then \( \phi_{n+1} \) can be selected by applying lemma 1.3 again. This proves the corollary. \( \square \)

We need one technical lemma whose proof can be seen in [7], namely;

**Lemma 1.5.** Let \( (f_n)_{n=1}^\infty \) be a sequence of nonnegative elements in \( wL_1 \) such that the \( f_n \) have pairwise disjoint supports with \( \|f_n\|_{wL_1} = 1 \), for all \( n = 1, 2, 3, \ldots \) and let \( (\phi_n)_{n=1}^\infty \) be a sequence of linear functionals on \( wL_1 \) selected as in Corollary 1.4. Then for any \( f \in wL_1 \), we have \( \sum_{n=1}^\infty |\phi_n(f)| \leq \|f\|_{wL_1} \).

**Proof.** See the lemma 1.6 in [7] \( \square \)

2. Complemented Banach subspaces in \( wL_1 \).

We study some relations between a Banach space \( E \) and \( wL_1 \). As an universal Banach lattice \( wL_1 \) for the separable Banach lattices, one can observe that \( wL_1 \) is a "big" Banach space. One can obviously ask the following questions:

What kind of Banach spaces can be embedded into \( wL_1 \)?

Moreover, is the range of the embedding map a complemented subspace of \( wL_1 \)?

We now start with the following proposition about the universality of \( wL_1 \).

**Proposition 2.1.** Every separable Banach space \( E \) is isometric to a Banach subspace of \( wL_1 \).

**Proof.** Let \( E \) be a separable Banach space. Then \( E \) is isometric to a subspace of \( C([0, 1]) \). Since \( C([0, 1]) \) is a separable Banach lattice with weak unit, by Lotz-Peck's theorem in [6], one can find a lattice isometry \( T : C[0, 1] \to wL_1 \) so that \( C[0, 1] \) is isometric and order isomorphic to a sublattice of \( wL_1 \). Then the restriction to \( E \) is the desired isometric embedding map. This proves the proposition. \( \square \)

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REMARK 2.2. Is every separable Banach subspace of $wL_1$ complemented? Unfortunately, the answer for this question is negative, by the theorem of Lindenstrauss and Tzafriri which states that if every closed subspace of given Banach space $E$ is complemented in $E$, then $E$ is isomorphic to Hilbert space. (As we know $wL_1$ is not a Hilbert space, moreover it is not a reflexive space.)

Nonetheless, one can find a lot of complemented Banach subspaces of $wL_1$. As one easy example, we give the following theorem.

**Theorem 2.3.** Let $(f_n)_{n=1}^\infty$ be a sequence in $wL_1$ such that the $f_n$ are nonnegative pairwise disjointly supported. If $E = \overline{\text{span}}(f_n)$, then $E$ is a complemented Banach subspace of $wL_1$.

**Proof.** Without loss of generality, we can assume that $\|f_n\|_{wL_1} = 1$, for all $n$. Hence by corollary 1.4, one can find linear functionals $\phi_n$ on $wL_1$ with $\phi_n(f_m) = \delta_{n,m}$, and $\|\phi_n\| = 1$ for all $n$. Now for arbitrary $f \in wL_1$, the number $\phi_n(f)$ is the limit of the subnet $\{I_U(\chi_{D_n,k} \cdot f)\}$ where $(D_{n,k})_{k=1}^\infty$ is a sequence of subsets of $D_n = \text{supp}(f_n)$.

Define $P : wL_1 \rightarrow E$ by

$$P(f) = \sum_{n=1}^{\infty} \phi_n(f)f_n.$$  

Moreover by the lemma 1.5, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_1}$. Hence $\|P\| \leq 1$.

Finally, we need to show that $P^2 = P$. For $f \in wL_1$,

$$P^2(f) = P\left(\sum_{n=1}^{\infty} \phi_n(f)f_n\right)$$

$$= \sum_{j=1}^{\infty} \phi_j\left(\sum_{i=1}^{\infty} \phi_i(f)f_i\right)f_j \quad \text{by} \quad \phi_i(f_j) = \delta_{ij}$$

$$= \sum_{j=1}^{\infty} \phi_j(f)f_j$$

$$= P(f)$$

Then $P$ is the desired projection on $wL_1$ with $\|P\| = 1$. This proves the theorem. \(\square\)
3. An application of Peck’s theorem

In this section, we give an easy application of Peck’s theorem, in [9], combined with well known Banach space results. Peck’s theorem gives some information about the decomposition of finite dimensional Banach subspaces of $wL_1$. (Especially, decompositions into $l_\infty^n$ and some other Banach space.) Considering [9], we state the theorem with no proof for our purpose.

**Theorem 3.1** (T. Peck). For $1 \leq i \leq n$, let $f_i > 0$ be elements of $wL_1$ such that the $f_i$ have pairwise disjoint supports, and such that $\|f_i\|_{wL_1} = 1$ for each $i$. Then there exist functions $e_i$, $0 \leq e_i \leq f_i$ so that

i) $\|\sum_{i=1}^{n} a_i e_i\|_{wL_1} = \sup_{1 \leq i \leq n} |a_i|$, for all $(a_i)_{i=1}^{n}$,

ii) if we set $g_i = f_i - e_i$, then $\|\sum_{i=1}^{n} a_i g_i\|_{wL_1} = \|\sum_{i=1}^{n} a_i f_i\|_{wL_1}$ for all $(a_i)_{i=1}^{n} \in \mathbb{R}^n$.

**Remark 3.2.** From this theorem, we can ask one obvious question. Namely, if we take away much more from each $f_i$, can we make the span of $(e_i)$ different from $l_\infty^n$, something like $l_p^n$? But these questions are still open.

Now fix an ultrafilter $\mathcal{F} \subset \mathcal{U}$ defined in (1.3). Define $L(\mathcal{U}) = \{f \in wL_1 : \|f\|_{wL_1} = \|f\|_\mathcal{U}\}$. In [4], one can see several properties of $L(\mathcal{U})$. Note that $\|f\|_\mathcal{U} = \mathcal{U}(|f|)$ in (1.5). We now are in a position to give some properties for $L(\mathcal{U})$.

**Proposition 3.3.** The set $L(\mathcal{U})$ is a norm closed subset of $wL_1$.

*Proof.* Let $(f_n)$ be a sequence such that $f_n$ converges to $f$ in $wL_1$. Then for all $\epsilon > 0$, there exists $N_0$ such that if $n \geq N_0$, we have $\|f_n - f\|_{wL_1} < \frac{\epsilon}{2}$. Now,
\[ \| f \|_U \leq \| f \|_{wL_1} \quad \text{by the theorem 1.1} \]
\[ \leq \| f - f_{N_0} \|_{wL_1} + \| f_{N_0} \|_{wL_1} \]
\[ \leq \frac{\epsilon}{2} + \| f_{N_0} \|_U \quad \text{by the definition of } L(U) \]
\[ = \frac{\epsilon}{2} + \| f_{N_0} \|_U - \| f \|_U + \| f \|_U \]
\[ \leq \frac{\epsilon}{2} + \| f - f_{N_0} \|_U - \| f \|_U + \| f \|_U \]
\[ \leq \frac{\epsilon}{2} + \| f - f_{N_0} \|_{wL_1} + \| f \|_U \]
\[ \leq \epsilon + \| f \|_U. \]

Since \( \epsilon > 0 \) is arbitrary, we have \( \| f \|_U \leq \| f \|_{wL_1} \leq \| f \|_U \). This implies \( f \in L(U) \). This proves the proposition. \( \square \)

**Examples.** We can give a lot of elements in \( L(U) \) as following. For each \( 0 \leq \alpha < 1 \), \( f(x) = \frac{1}{x - \alpha} \) is an element of \( L(U) \).

\[ 1 = \| f \|_{wL_1} \]
\[ = \| f \|_U. \]

Hence these examples and the proposition 3.3 give that \( L(U) \) contains infinite number of elements. Next we give one more property of \( L(U) \). That is, every finite sequence of pairwise disjointly supported nonnegative elements in \( L(U) \) form a basis of \( l_1 \).

**Theorem 3.4.** Let \( f_1, f_2, \ldots, f_n \) be pairwise disjointly supported nonnegative elements in \( L(U) \) with \( \| f_i \|_{wL_1} = 1 \), for all \( i = 1, 2, 3, \ldots, n \). Then the span of \( (f_i)_{i=1}^n \) in \( wL_1 \) is isometrically isomorphic to \( l_1^n \).

**Proof.** Define \( T : l_1^n \rightarrow \text{span}(f_i) \), by \( Te_i = f_i \) where \( (e_i)_{i=1}^n \) is the usual basis of \( l_1^n \). Then \( T(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i f_i \) is a well defined continuous linear operator.

We need to show that \( \| \sum_{i=1}^n a_i f_i \|_{wL_1} = \sum_{i=1}^n |a_i| \). Since \( \| \cdot \|_{wL_1} \) is a lattice norm, one can assume with no loss of generality \( a_i > 0 \) and
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$a_i \neq 0$, for all $i$. Now,

$$
\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_i \|f_i\|_{wL_1} \\
= \sum_{i=1}^{n} \|a_i f_i\|_{wL_1} \\
= \sum_{i=1}^{n} a_i \|f_i\|_U \quad (\|f_i\|_{wL_1} = \|f_i\|_{wL_1}, \forall i) \\
= \sum_{i=1}^{n} \|a_i f_i\|_U \\
= \| \sum_{i=1}^{n} a_i f_i \|_U \quad \text{(by the property of } \| \cdot \|_U) \\
\leq \| \sum_{i=1}^{n} a_i f_i \|_{wL_1} \quad \text{by the theorem 1.1} \\
\leq \sum_{i=1}^{n} a_i \|f_i\|_{wL_1} = \sum_{i=1}^{n} a_i.
$$

Therefore $T$ is linear isometry from $l_1$ onto $\overline{\text{span}}(f_i)$. This proves the theorem. ☐

**Corollary 3.5.** Let $(f_i)_{i=1}^{\infty}$ be a sequence of nonnegative elements in $L(U)$ which have pairwise disjoint supports, with each $\|f_i\|_{wL_1} = 1$. Then $\overline{\text{span}}(f_i)_{i=1}^{\infty}$ is isometrically isomorphic to $l_1$.

**Proof.** Define $T : l_1 \rightarrow \overline{\text{span}}(f_i)_{i=1}^{\infty}$ by $Te_i = f_i$, for all $i$. Then $T$ is linear and $T(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} a_i f_i$. Exactly the same argument in the theorem 3.4 gives an isometric isomorphism of $T$. This proves the corollary. ☐

Next, using Peck's theorem we can give one more property of the envelope norm in $wL_1$. For this, we need a technical lemma.
Lemma 3.6. For $h \in \mathcal{W}_{1}$, $h \geq 0$, $h$ can be written as a strictly increasing nonnegative function with no atoms.

Proof. Let $h \in \mathcal{W}_{1}$, $\|h\|_{\mathcal{W}_{1}} = 1$ and $h \geq 0$. If the function $h$ satisfies $\mu(h = r) > 0$ for some $r$, then we embed the measure algebra of Lebesgue measure $\lambda$ on $[0, 1]$ into the measure algebra of normalized $\mu$-measure on the measurable set $\{h = r\}$. We replace $h$ on this set by the image of the function $\phi(t) = t + r$. Since there are at most countably many points $r \geq 0$ for which $\mu\{h = r\} > 0$, the performance of such replacement for each of these points will change $h$ by at most a bounded measurable function. This means that $h$ can be everywhere strictly positive. This proves the lemma. \qed

Now, if we use one element $f \in \mathcal{W}_{1}$, then we can get the following theorem.

Theorem 3.7. Let $f \in \mathcal{W}_{1}$ be a nonnegative element with $\|f\|_{\mathcal{W}_{1}} = 1$. Then there exists a sequence $(E_{n})_{n=1}^{\infty}$ of subspaces of $\mathcal{W}_{1}$ such that

i) $E_{1} = \text{span}(f)$,

ii) $\dim(E_{n}) = n$, for all $n = 1, 2, 3, \ldots$,

iii) $E_{n}$ is isometrically isomorphic to $l_{\infty, n}$, for all $n = 1, 2, 3, \ldots$.

Proof. Assume $\|f\|_{\mathcal{W}_{1}} = 1$. Then there exist disjoint intervals $\{[a_{n}, b_{n}]\}_{n=1}^{\infty}$ such that $b_{n}/a_{n} \to \infty$ and $a_{n} \to \infty$,

\begin{equation}
\frac{1}{\log \frac{b_{n}}{a_{n}}} \int_{\{a_{n} \leq x \leq b_{n}\}} f(x) d\mu \geq 1 - \frac{1}{n}, \quad \text{for all } n = 1, 2, 3, \ldots
\end{equation}

Without loss of generality, we can assume $f$ is everywhere strict by the lemma 3.6. The construction of the subspaces $E_{n}$ follow an inductive argument by using Peck's theorem. Let $E_{1} = \text{span}(f)$. Then $\dim(E_{1}) = 1$.

We now construct a 2-dimensional subspace $E_{2}$ of $\mathcal{W}_{1}$ satisfying the conclusion of the theorem. Let

\begin{equation}
e_{1} = \sum_{n=1}^{\infty} f \chi_{(a_{2n} \leq x \leq b_{2n})}.
\end{equation}
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Then clearly $\|e_1\|_{wL_1} = 1$. And let $g_1 = f - e_1$. Then by theorem 3.1, we have $\|g_1\|_{wL_1} = \|f - e_1\|_{wL_1} = \|f\|_{wL_1} = 1$. Hence we have $g_1$ and $e_1$ which are nonnegative and disjointly supported with $\|e_1\|_{wL_1} = \|g_1\|_{wL_1} = 1$. Let $E_2$ be the subspace of $wL_1$ generated by $\{g_1, e_1\}$. Then $\dim(E_2) = 2$ and $E_2$ is constructed from $E_1$ in this sense. Define $T_2 : l^2_\infty \rightarrow E_2$ by $T_2(a_i) = a_1g_1 + a_2e_1$. Then $T_2$ is an isomorphism from $l^2_\infty$ onto $E_2$. To show $T_2$ is an isometry, it suffices to show that $\|g_1 + e_1\|_{wL_1} = 1$, since $\|\cdot\|_{wL_1}$ is a lattice norm on $wL_1$. But this holds, since $\|g_1 + e_1\|_{wL_1} = \|f\|_{wL_1} = 1$. Hence we have

$$
\|T_2(a_i)\|_{wL_1} = \|a_1g_1 + a_2e_1\|_{wL_1} = \|(a_i)\|_{\infty}.
$$

(3.3)

Next, we give the argument for constructing $E_3$ from $E_2$ satisfying the conclusion of theorem. Then step by step, by applying the same argument, we can construct $E_n$ from $E_{n-1}$. Fix $e_1$ in $E_2$. Decompose $g_1$ as $g_2$ and $e_2$ using (3.1) and Peck’s theorem. Then we have three pairwise disjointly supported elements $\{g_2, e_1, e_2\}$ with $\|e_1\|_{wL_1} = \|g_2\|_{wL_1} = \|e_2\|_{wL_1} = 1$. Then $\text{span}(g_2, e_1, e_2) = E_3$. Then $\dim(E_3) = 3$. The rest of the argument about isometry exactly follows the previous one. This proves the theorem.

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