CLIFFORD $L^2$-COHOMOLOGY ON THE COMPLETE KÄHLER MANIFOLDS II

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ABSTRACT. In this paper, we prove that on the complete Kähler manifold, if $\rho(x) \geq -\frac{1}{2}\lambda_0$ and either $\rho(x_0) > -\frac{1}{2}\lambda_0$ at some point $x_0$ or $\text{Vol}(M) = \infty$, then the Clifford $L^2$-cohomology group $L^2\mathcal{H}^*(M, S)$ is trivial, where $\rho(x)$ is the least eigenvalue of $\mathcal{R}_x + \mathcal{R}(x)$ and $\lambda_0$ is the infimum of the spectrum of the Laplacian acting on $L^2$-functions on $M$.

0. One of the important object in the study of a manifold is its Clifford algebra $Cl(M)$, generated by the tangent space. It carries an intrinsic first order elliptic operator $D$, which is called the Dirac operator. There is a canonical vector (but not algebra) bundle isomorphism $\Lambda^*(M) \rightarrow Cl(M)$, where $\Lambda^*(M)$ is an exterior algebra of $M$. In $\Lambda^*(M)$, the Dirac operator $D$ is $D \cong d + \delta$ and the Laplace operator is the square of the Dirac operator, where $d$ is the exterior differential and $\delta$ is the adjoint operator of $d$. Therefore many results of the Clifford theory yield the results of the de Rham theory ([8]). In 1980, M. L. Michelsohn ([10]) proved many results for the Dirac operator on compact Kähler manifold. Recently, J. S. Pak and S. D. Jung ([11]) extended the results of M. L. Michelsohn ([10]) and obtained the following theorem for the Dirac operator on complete Kähler manifold.

THEOREM A. Let $M$ be a complete Kähler manifold and $S$ be any hermitian vector bundle of modules over $Cl(M)$. If $R$ is non-negative and positive at some point of $M$, then the Clifford $L^2$-cohomology group

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is trivial, where $R$ is the symmetric endomorphism of $S$ containing the curvature data.

In this paper, we prove Theorem A under the assumption of weaker curvature endomorphism $R$ which is bounded by $-\frac{1}{2} \lambda_0$ from below, $\lambda_0$ is the infimum of the spectrum of the positive Laplacian $\Delta^M$ acting on $L^2$-functions on $M$. The method of this study is based on that of P. Bérard ([2]). From our results, we deduce the vanishing theorem for the harmonic forms which extend the results of K. D. Elworthy and S. Rosenberg ([4]) to the Kähler case. Also, we study the harmonic spinors under some condition of the scalar curvature.

1. Let $M$ be a $2n$-dimensional Kähler manifold with almost complex structure $J$ and with connection $\nabla$. Let $Cl(M)$ be the Clifford bundle generated by the tangent bundle $TM$. Now we define a derivation $J_0 : Cl(M) \to Cl(M)$ induced by $J$ as follows:

\[
J_0(v_1 \cdots v_k) = \sum_{j=1}^{k} v_1 \cdots J v_j \cdots v_k
\]

for $v_1, \cdots, v_k \in TM$, where "\cdots" is the Clifford multiplication. If it is clear from the context which multiplication is meant, we omit the Clifford multiplication "\cdot". To study $J_0$ effectively we consider the complexification $Cl(M) = Cl(M) \otimes_{\mathbb{R}} \mathbb{C}$. This algebra has a natural basis given as follows: Let $e_1, \cdots, e_n, J e_1, \cdots, J e_n$ be an orthonormal basis of $T_x M$. Let $T_x^{1,0}$ (resp. $T_x^{0,1}$) be the $i$ eigenvalue (resp. $-i$ eigenvalue) of $J$ in $T_x M \otimes \mathbb{C}$. Put

\[
\xi_k = \frac{1}{2} \{ e_k - i J e_k \}, \quad \bar{\xi}_k = \frac{1}{2} \{ e_k + i J e_k \}.
\]

Then $\xi_1, \cdots, \xi_n$ (resp. $\bar{\xi}_1, \cdots, \bar{\xi}_n$) is the basis of $T_x^{1,0}$ (resp. $T_x^{0,1}$). And \{\xi_k, \bar{\xi}_k\} has the following properties;

\[
(1.2) \quad \xi_k \bar{\xi}_e + \bar{\xi}_k \xi_e = \xi_k \bar{\xi}_e + \bar{\xi}_e \xi_k = -\delta_{k e}, \quad \xi_k \xi_e = -\xi_e \xi_k, \quad \bar{\xi}_k \bar{\xi}_e = -\bar{\xi}_e \bar{\xi}_k.
\]

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Denote $\xi_K \tilde{\xi}_I = \xi_{k_1} \ldots \xi_{k_r} \tilde{\xi}_{i_1} \ldots \tilde{\xi}_{i_s}$, where $K$ and $I$ range over all strictly ascending multiindices from $\{1, \ldots, n\}$. For convenience we set $\mathcal{J} = \frac{1}{i} \mathcal{J}_0$. Then by the derivation property, we have

$$\mathcal{J}(\xi_K \tilde{\xi}_I) = (|K| - |I|)\xi_K \tilde{\xi}_I,$$

where $|K|, |I|$ denote the lengths of $K$ and $I$. This gives a decomposition

$$\text{Cl}(M) = \bigoplus_{p=-n}^{n} \text{Cl}^p(M),$$

where $\text{Cl}^p(M) = \{\phi \in \text{Cl}(M) \mid \mathcal{J}\phi = p\phi\}$.

We now introduce two intrinsically defined linear maps $\mathcal{L}, \tilde{\mathcal{L}} : \text{Cl}(M) \to \text{Cl}(M)$ as follows; For any $\varphi \in \text{Cl}(M)$, set

$$\mathcal{L}(\varphi) = -\sum_{k=1}^{n} \xi_k \varphi \tilde{\xi}_k, \quad \tilde{\mathcal{L}}(\varphi) = -\sum_{k=1}^{n} \tilde{\xi}_k \varphi \xi_k.$$

These operators are independent of the Hermitian basis chosen to define them. We consider the operator $\mathcal{H} = [\mathcal{L}, \tilde{\mathcal{L}}]$. Then they satisfy the following relations;

$$[\mathcal{L}, \tilde{\mathcal{L}}] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{L}] = 2\mathcal{L}, \quad [\mathcal{H}, \tilde{\mathcal{L}}] = -2\tilde{\mathcal{L}}.$$

Hence they define a representation of $sl(2, \mathbb{C})$, the Lie algebra of $SL(2, \mathbb{C})$ on $\text{Cl}(M)$. Since each of the operators $\mathcal{L}, \tilde{\mathcal{L}}$ and $\mathcal{H}$ commutes with $\mathcal{J}$, we can define the subspaces

$$\text{Cl}^{p,q}(M) = \{\varphi \in \text{Cl}(M) \mid \mathcal{H}\varphi = q\varphi, \mathcal{J} = p\varphi\}$$

and obtain a decomposition ([10])

$$\text{Cl}(M) = \bigoplus_{p,q} \text{Cl}^{p,q}(M).$$
PROPOSITION 1.1 ([10]). For each $\xi \in T^{1,0}(M)$, one has that $\xi \cdot \mathbb{Cl}_{p,q} \subset \mathbb{Cl}_{p+1,q+1}$ and $\bar{\xi} \cdot \mathbb{Cl}_{p,q} \subset \mathbb{Cl}_{p-1,q-1}$. Furthermore, if $\xi \neq 0$, the sequences

$$
\ldots \xrightarrow{\lambda_\xi} \mathbb{Cl}_{p-1,q-1} \xrightarrow{\lambda_\xi} \mathbb{Cl}_{p,q} \xrightarrow{\lambda_\xi} \mathbb{Cl}_{p+1,q+1} \xrightarrow{\lambda_\xi} \mathbb{Cl}_{p-1,q-1} \xrightarrow{\lambda_\xi} \mathbb{Cl}_{p,q} \xrightarrow{\lambda_\xi} \mathbb{Cl}_{p+1,q+1} \ldots
$$

where $\lambda_\xi$ denotes left Clifford multiplication by $\xi$, are exact.

2. Suppose that $M$ is a complete Kähler manifold. We introduce two differential operators $\mathcal{D}, \bar{\mathcal{D}} : \Gamma\mathbb{Cl}(M) \to \Gamma\mathbb{Cl}(M)$ by the formulas

$$(2.1) \quad \mathcal{D} = \sum_j \xi_j \nabla \xi_j, \quad \bar{\mathcal{D}} = \sum_j \bar{\xi}_j \nabla \bar{\xi}_j,$$

where $\nabla$ is the canonical connection. Since $\nabla$ preserves the subbundles $\Gamma\mathbb{Cl}_{p,q}(M)$, we have

$$\mathcal{D}(\Gamma\mathbb{Cl}_{p,q}) \subset \Gamma\mathbb{Cl}_{p+1,q+1}, \quad \bar{\mathcal{D}}(\Gamma\mathbb{Cl}_{p,q}) \subset \Gamma\mathbb{Cl}_{p-1,q-1}$$

for all $p$ and $q$. Then we have the following well known fact:

THEOREM 2.1 ([10]). The operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoints of one another on $\Gamma_{\text{cpt}}\mathbb{Cl}(M)$, the set of all sections with the compact support. And they satisfy

$$\mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0.$$

Furthermore, the complex

$$
\ldots \xrightarrow{\mathcal{D}} \Gamma\mathbb{Cl}_{p-1,q-1} \xrightarrow{\mathcal{D}} \Gamma\mathbb{Cl}_{p,q} \xrightarrow{\mathcal{D}} \Gamma\mathbb{Cl}_{p+1,q+1} \xrightarrow{\mathcal{D}} \ldots
$$

is elliptic.

Now we set

$$(2.2) \quad \Delta := \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}.$$
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Then $\Delta$ is a formally self-adjoint elliptic operator. To understand $\Delta$ we introduce two "real" operators on $\text{Cl}(M)$:

$$(2.3)\quad D = \sum_j \{e_j \nabla e_j + (Je_j) \nabla e_j\}, \quad D^c = \sum_j \{e_j \nabla e_j - (Je_j) \nabla e_j\}.$$  

The first operator is called the Dirac operator. Then we can easily see that

$$(2.4)\quad D = \frac{1}{4}(D + iD^c), \quad \bar{D} = \frac{1}{4}(D - iD^c).$$

Since $D^2 = 0$, we have that $D^2 = (D^c)^2$ and $DD^c + D^cD = 0$. It follows that

$$(2.5)\quad \Delta = \frac{1}{4}D^2.$$  

Since $D$ is essentially self-adjoint, we have

$$(2.6)\quad \text{Ker} D = \text{Ker} D^2 = \text{Ker} \Delta.$$  

Now, we consider the usual inner product

$$(2.7)\quad \langle \langle \varphi_1, \varphi_2 \rangle \rangle = \int_M \langle \varphi_1, \varphi_2 \rangle$$

for any $\varphi_1, \varphi_2 \in \Gamma_{\text{cpt}}\text{Cl}(M)$. Let $L^2(\text{Cl}^{p,q}(M))$ be the completion of $\Gamma_{\text{cpt}}\text{Cl}^{p,q}$ with respect to $\langle \langle , \rangle \rangle$. We recall that the operators $D$ and $\bar{D}$ are formal adjoint to one another with respect to $\langle \langle , \rangle \rangle$. Then $D$ and $\bar{D}$ have closed extensions in $L^2(\text{Cl}^{p,q}(M))$. But since $M$ is complete, their closed extensions are unique ([3]). From now on, we write the closed extensions as the same symbols. Now, we put

$$(2.8)\quad L^2\mathcal{H}^{p,q} := \text{Ker} D / \text{Im} \bar{D} \cap L^2(\text{Cl}^{p,q}(M)),$$

$$(2.9)\quad L^2\hat{\mathcal{H}}^{p,q} := \text{Ker} D \cap \text{Ker} \bar{D} \cap L^2(\text{Cl}^{p,q}(M)),$$

$$(2.10)\quad L^2H^{p,q} := \text{Ker} \Delta \cap L^2(\text{Cl}^{p,q}(M)).$$

Here $L^2\mathcal{H}^{p,q}$ and $L^2H^{p,q}$ are called the Clifford $L^2$-cohomology group and $L^2$-harmonic space, respectively. Then we have
PROPOSITION 2.2 ([11]). Let $M$ be a complete Kähler manifold. Then we have
\[ L^2 \mathcal{H}^{p,q} \cong L^2 \tilde{\mathcal{H}}^{p,q} \cong L^2 \mathcal{H}^{p,q}. \]

3. Let $M$ be a Kähler manifold and $S \to M$ a hermitian vector bundle of left modules over $\mathcal{C}l(M)$ with a hermitian metric $\langle \cdot, \cdot \rangle$ such that:

(1) Module multiplication by unit tangent vectors is unitary, i.e.,
\[ \langle \xi \cdot \phi, \psi \rangle + \langle \phi, \xi \cdot \psi \rangle = 0, \]
for any $\phi, \psi \in \Gamma(S)$ and $\xi \in \Gamma(TM) \otimes \mathbb{C}$

(2) With respect to the canonical hermitian connection, covariant differentiation is a derivation of module multiplication. That is, for $\phi \in \Gamma(\mathcal{C}l(M))$ and $s \in \Gamma(S)$, we have
\[ \nabla(\phi \cdot s) = (\nabla \phi) \cdot s + \phi \cdot (\nabla s). \]

Now, we recall some basic results from [10]. For each $j$, we set $\omega_j = -\xi_j \bar{\xi}_j$, $\bar{\omega}_j = -\bar{\xi}_j \xi_j$. To each (possibly empty) subset $I = \{i_1, \cdots, i_p\} \subseteq \{1, \cdots, n\}$ with complementary subset $\{j_1, \cdots, j_{n-p}\}$ we set $\omega_I = \omega_{i_1} \cdots \omega_{i_p} \bar{\omega}_{j_1} \cdots \bar{\omega}_{j_{n-p}}$ and we denote $|I| = p$. Then we have
\[ 1 = \prod_{j=1}^{n} (\omega_j + \bar{\omega}_j) = \sum_{r=1}^{n} \pi_r, \]
where $\pi_r = \sum_{|I|=r} \omega_I$. Moreover, we have an orthogonal decomposition of the bundle
\[ S = \bigoplus_{r=0}^{n} S^r, \quad S^r = \pi_r \cdot S. \]

Then the complex
\[ 0 \to \Gamma_{cpt}(S^0) \overset{D}{\to} \Gamma_{cpt}(S^1) \overset{D}{\to} \cdots \overset{D}{\to} \Gamma_{cpt}(S^n) \to 0 \]
is elliptic and its completion becomes a Hilbert complex ([3]). Similarly with Proposition 2.2, we have

\[ L^2\mathcal{H}^r(M, S) \cong L^2\mathcal{H}^r(M, S) \cong L^2H^r(M, S). \]

Now, we define invariant operators on $\Gamma(S)$ by

\[ \nabla^* \nabla = -\sum_j \nabla_{\xi_j, \xi_j}, \quad \tilde{\nabla}^* \tilde{\nabla} = -\sum_j \nabla_{\tilde{\xi}_j, \tilde{\xi}_j}, \]

\[ \mathcal{R} = \sum_{j,k} \xi_j \xi_k R_{\xi_j, \xi_k}, \quad \tilde{\mathcal{R}} = \sum_{j,k} \tilde{\xi}_j \tilde{\xi}_k R_{\tilde{\xi}_j, \tilde{\xi}_k}, \]

where $R_{V,W} = \nabla_{V,W} - \nabla_{W,V}$ is the curvature tensor and where $\nabla_{V,W} = \nabla_V \nabla_W - \nabla_{\nabla_V W}$ is the invariant second covariant derivative. Then we obtain

**Proposition 3.1** ([10]). For any two sections $s_1, s_2 \in \Gamma(S)$, at least one of which has compact support, the following holds:

\[ \int_M \langle \nabla^* \nabla s_1, s_2 \rangle = \int_M \langle \nabla s_1, \nabla s_2 \rangle, \]

where $\langle \nabla s_1, \nabla s_2 \rangle = \langle \nabla_{\xi_1} s_1, \nabla_{\xi_2} s_2 \rangle$. Hence $\nabla^* \nabla$ is a formally self adjoint, nonnegative operator. Similarly, this holds for $\tilde{\nabla}^* \tilde{\nabla}$. Moreover, the zero order operators $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are self-adjoint.

By the straight calculation, we obtain the Bochner-Weitzenböck type formula ([10]);

\[ \mathcal{D} \tilde{\mathcal{D}} + \tilde{\mathcal{D}} \mathcal{D} = \nabla^* \nabla + \mathcal{R} = \tilde{\nabla}^* \tilde{\nabla} + \tilde{\mathcal{R}}. \]

From this formula, we have

\[ 2(\mathcal{D} \tilde{\mathcal{D}} + \tilde{\mathcal{D}} \mathcal{D}) = \nabla^* \nabla + \tilde{\nabla}^* \tilde{\nabla} + \mathcal{R} + \tilde{\mathcal{R}}. \]

Let $\rho(x)$ denote the least eigenvalue of $R_x(= \mathcal{R}_x + \tilde{\mathcal{R}}_x)$, the symmetric endomorphism of $S_x$, that is,

\[ \rho(x) = \inf \{(R_x(s), s)_{S_x} \mid s \in S_x, \ |s| = 1\} \]

and $\lambda_0$ is the infimum of the spectrum of the positive Laplacian $\Delta^M$ acting on $L^2$-functions on $M$, that is, $\Delta^M = \delta d$, where $\delta$ is the adjoint operator of $d$. Then we have
THEOREM 3.3. Let $M$ be a complete Kähler manifold and let $S$ be any hermitian vector bundle of modules over $\mathcal{C}l(M)$. If $\rho(x) \geq -\frac{1}{2}\lambda_0$ for all $x \in M$ and either $\rho(x_0) > -\frac{1}{2}\lambda_0$ for some $x_0 \in M$ or $(M, g)$ has infinite volume, then the Clifford $L^2$-cohomology group is trivial. That is,$$
abla^2 M^\tau(M, S) = \{0\}, \quad \text{for any } r = 0, 1, \cdots, n.$$

In order to prove that Theorem 3.3, we prepare some Lemmas;

LEMMA 3.4 ([2]) (the first Kato inequality). For any $s \in \Gamma(S)$, $|ds| \leq |\nabla s|$, with equality if and only if for any $X \in TM$, there exists a function $f_X$ such that $\nabla_X s = f_X s$ (at least on the set $\{|s| \neq 0\}$).

LEMMA 3.5 ([2]). If $s \in \Gamma(S)$ satisfies $|ds| = |\nabla s|$, then on $\{s \neq 0\}$, $s = |s|s_1$, with $\nabla s_1 = 0$.

LEMMA 3.6 ([2]) (the second Kato inequality). If $s \in \Gamma(S)$ satisfies $\Delta s = 0$, then $\Delta^M |s| \leq -2\rho |s|$ with equality if and only if $|ds| = |\nabla s|$ and $\langle R(s), s \rangle = 2\rho |s|^2$, where $R = \mathcal{R} + \mathcal{R}$.

Proof of Theorem 3.3. By (3.6), it is sufficient to prove that $L^2 H^\tau(M, S) = \{s \in \text{Ker} \Delta |s| \in L^2(M, S^r)\} = \{0\}$. This proof is based on the method of P. Bérard ([2]). Let $s \in \text{Ker} \Delta$ of finite $L^2$-norm and denote $\phi := |s|$, its pointwise norm. First, we assume that $\rho(x) \geq -\frac{1}{2}\lambda_0$ for all $x \in M$. Using Lemma 3.6, we have

\begin{equation}
\Delta^M \phi \leq -2\rho \phi \leq \lambda_0 \phi.
\end{equation}

Since $M$ is complete, one can construct function $\omega_\ell$ such that $\omega_\ell \in C^\infty_0(M)$ and $\omega_\ell \equiv 1$ on $B(x_0, \ell)$, supp $\omega_\ell \subset B(x_0, 2\ell)$ and $|d\omega_\ell| \leq C/\ell$ for some constant $C$, where $\ell \in \mathbb{R}_+$, $x_0 \in M$ and $B(x_0, \ell)$ is the Riemannian open ball with radius $\ell$ and center $x_0$. Multiplying (3.10) by $\omega^2_\ell \phi$ and integrating by parts, we obtain

\begin{equation}
\int \langle d\phi, d\omega^2_\ell \phi \rangle \leq -2 \int \rho \omega^2_\ell \phi^2 \leq \lambda_0 \int \omega^2_\ell \phi^2,
\end{equation}

where $\langle \ , \ \rangle$ denotes the hermitian metric on $T^*M$. By straight calculation, we have the equality

\begin{equation}
\int \omega^2_\ell |d\phi|^2 + 2 \int \omega_\ell \phi \langle d\omega_\ell, d\phi \rangle = \int |d(\omega_\ell \phi)|^2 - \int \phi^2 |d\omega_\ell|^2.
\end{equation}
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Summing (3.11) and (3.12), we obtain

$$
\int |d(\omega \phi)|^2 \leq -2 \int \rho \omega^2 \phi^2 + \int \phi^2 |d\omega^2| \leq \lambda_0 \int \omega^2 \phi^2 + \int \phi^2 |d\omega^2|.
$$

On the other hand, since $\lambda_0$ is the infimum of the spectrum of $\Delta^M$, we get

$$
\int |d(\omega \phi)|^2 \geq \lambda_0 \int (\omega \phi)^2.
$$

From (3.13) and (3.14), we get

$$
\lambda_0 \int (\omega \phi)^2 \leq \int \phi^2 |d\omega| - 2 \int \rho \omega^2 \phi^2 \leq \lambda_0 \int \omega^2 \phi^2 + \int \phi^2 |d\omega^2|.
$$

Now, if we let $\ell \to \infty$, then by the property $|d\omega^2| \leq \frac{C}{\ell}$, we obtain

$$
\lambda_0 \int \phi^2 \leq -2 \int \rho \phi^2 \leq \lambda_0 \int \phi^2.
$$

Under the assumption $\rho(x_0) > -\frac{1}{2} \lambda_0$ for some $x_0$, this implies that $\phi = 0$.

Now, we prove the second part. From the inequality $|2(a, b)| \leq t^2 |a|^2 + \frac{1}{t^2} |b|^2$ for any $t \in \mathbb{R}$, we have

$$
|2 \int \omega \phi (d\phi, d\omega^2)| \leq t^2 \int \omega^2 |d\phi|^2 + \frac{1}{t^2} \int \phi^2 |d\omega^2|.
$$

Comparing (3.12), (3.14) and (3.16), we obtain

$$
(1 - t^2) \int \omega^2 |d\phi|^2 \leq -2 \int \rho \omega^2 \phi^2 + \frac{1}{t^2} \int \phi^2 |d\omega^2|
\leq \lambda_0 \int \omega^2 \phi^2 + \frac{1}{t^2} \int \phi^2 |d\omega^2|.
$$

Taking $t = \ell^{-\frac{1}{2}}$ and letting $\ell \to \infty$, the above inequality becomes

$$
\int |d\phi|^2 \leq -2 \int \rho \phi^2 \leq \lambda_0 \int \phi^2.
$$
and hence \( \phi \in \mathcal{S}^1(M) \) (=the first sobolev space). Similarly from (3.16), we obtain the inequality

\[
(1 + t^2) \int \omega_\phi^2 |d\phi|^2 \geq \int |d(\omega \phi)|^2 - (1 + \frac{1}{t^2}) \int \phi^2 |d\omega|^2 \\
\geq \lambda_0 \int \omega_\phi^2 \phi^2 - (1 + \frac{1}{t^2}) \int \phi^2 |d\omega|^2.
\]

Taking \( t = \ell^{-\frac{1}{2}} \) and letting \( \ell \to \infty \), we get

\[(3.18) \quad \int |d\phi|^2 \geq \lambda_0 \int \phi^2.\]

From (3.17) and (3.18), we have \( \int |d\phi|^2 = \lambda_0 \int \phi^2 \). Since \( \lambda_0 \) is the infimum of the spectrum of \( \Delta^M \), we have \( \Delta^M \phi = \lambda_0 \phi \) which implies that \( \phi \in C^\infty(M) \). By maximum principle and \( \phi \geq 0 \), \( \phi = 0 \) or \( \phi > 0 \) everywhere. Assume \( \phi \neq 0 \). By Lemma 3.4 and our assumption, \( \lambda_0 \phi = \Delta^M \phi \leq -2\rho \phi \leq \lambda_0 \phi \). That is, \( \Delta^M \phi = -2\rho|s| \). This implies that \( s = |s|s_1 \) with \( \nabla s_1 = 0 \) everywhere and \( \langle Rs_1, s_1 \rangle = -\lambda_0 \). Because \( \Delta s_1 = \nabla s_1 = 0 \), this implies that

\[-\lambda_0 = \langle Rs_1, s_1 \rangle = \langle (\mathcal{R} + \bar{\mathcal{R}})s_1, s_1 \rangle = 0 \]

and hence \( \phi \) is constant and \( s_1 \in L^2(S) \). Hence we have that if \( \text{Vol}(M) = +\infty \), then \( \phi = 0 \). \( \square \)

Moreover, on \( TM \subset \mathcal{C}l(M) \), we have ([8])

\[\mathcal{R} + \bar{\mathcal{R}} = \frac{1}{2} \text{Ric}.\]

Hence we have

**Corollary 3.7.** On the complete Kähler manifold, if \( \text{Ric} \geq -\lambda_0 \) and \( \text{Ric} > -\lambda_0 \) at some point \( x_0 \), then every \( L^2 \)-harmonic 1-form is necessarily zero.

4. We shall consider some special cases of the results above. To begin, we suppose that \( M \) is a Kähler spin manifold, i.e., we assume that
there exists a principal Spin-bundle, $P_{\text{Spin}}(M) \to M$, with a $\text{Spin}_{2n}$-equivalent map $\tau : P_{\text{Spin}}(M) \to P_{\text{SO}}(M)$, to the bundle of real oriented orthonormal frame on $M$. The bundle of spinors, $S$, is then defined to be vector bundle associated to the unitary representation $\Delta$ of $\text{Spin}_{2n}$ given by the unique irreducible complex representation of $\text{Cl}_{2n}$, i.e., $S = P_{\text{Spin}} \times_{\Delta} \mathbb{C}^{2n}$. This bundle is naturally a bundle of modules over $\text{Cl}(M)$ and carries a canonical connection induced from the lift of the riemannian connection on $P_{\text{SO}}(M)$. Since $M$ is Kähler, this bundle $S$ is naturally holomorphic and its connection is hermitian. On this bundle $S$, the curvature tensor $R^S$ is given by

\begin{equation}
R^S_{V,W} = \frac{1}{4} \sum_{\alpha, \beta = 1}^{2n} \langle R_{V,W} X_\alpha, X_\beta \rangle X_\alpha X_\beta,
\end{equation}

where $X_1, \cdots, X_{2n}$ is any real orthonormal basis of the tangent space ([8]). Choosing a basis $e_1, \cdots, J e_n$, we can write $R^S$ as

\begin{equation}
R^S_{V,W} = 2 \sum_{j,k=1}^{n} \langle R_{V,W} \xi_j, \bar{\xi}_k \rangle \xi_j \bar{\xi}_k + \sum_{j=1}^{n} \langle R_{V,W} \xi_j, \bar{\xi}_j \rangle.
\end{equation}

Hence we have

\begin{equation}
R^S = \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k R^S_{\xi_j, \xi_k}
= \sum_{i,j,k=1}^{n} \langle R_{\xi_i, \bar{\xi}_i \xi_j, \xi_k} \rangle \xi_j \bar{\xi}_k
= -\frac{1}{2} \sum_{j,k=1}^{n} \text{Ric}(\xi_j, \xi_k) \xi_j \bar{\xi}_k,
\end{equation}

where $\text{Ric}$ is Ricci tensor on $M$ ([10]). Since $\text{Ric}$ is hermitian symmetric, we may choose our basis so that $\text{Ric}(\xi_j, \xi_k) = 1/2 \lambda_j \delta_{jk}$, where $\lambda_j = \text{Ric}(e_j, e_j) = \text{Ric}(Je_j, Je_j)$, for $j = 1, \cdots, n$, are the eigenvalues. Then we have

\begin{equation}
\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \nabla^* \nabla + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \omega_j = \bar{\nabla}^* \bar{\nabla} + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \bar{\omega}_j.
\end{equation}
We note that $\nabla^* \nabla + \tilde{\nabla}^* \tilde{\nabla} = \frac{1}{2} \tilde{\nabla}^* \tilde{\nabla}$ where

\begin{equation}
\tilde{\nabla}^* \tilde{\nabla} = -\sum_j \left( \nabla_{e_j, e_j} + \nabla_{Je_j, Je_j} \right)
\end{equation}

is a self-adjoint, elliptic operator whose kernel is the space of parallel sections ([8]). We note that the scalar curvature $\kappa$ of $M$ is given by

\begin{equation}
\kappa = \text{trace}_R(\text{Ric}) = 2 \sum_j \lambda_j.
\end{equation}

Hence we get

**THEOREM 4.1 ([10]).** On the spinor bundle $S$, we have

$$4(DD + \tilde{D} \tilde{D}) = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} \kappa,$$

where $\kappa$ is the scalar curvature of $M$.

Summing up Theorem 3.3 and Theorem 4.1, we have

**THEOREM 4.2.** Let $M$ be a complete Kähler spin manifold. If $\kappa \geq -4\lambda_0$ for all $x \in M$ and either $\kappa > -4\lambda_0$ for some $x_0 \in M$ or $(M, g)$ has infinite volume, then there are no non-trivial $L^2$-harmonic spinors.

**References**

Clifford $L^2$-cohomology


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