THE $C^r$ CLOSING LEMMA FOR CHAIN
RECURRANCE IN COMPACT SURFACES

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ABSTRACT. We prove the $C^r$ Closing Lemma for chain recurrence
in compact surfaces.

Solving the Closing Lemma is interesting in its own right, but more
because it implies generically that a dynamical system already has its
periodic orbits dense in its set of nonwandering orbits. The first state-
ment and proof of $C^1$ Closing Lemma (for nonwandering points) are
due to Pugh [5]. There was a gap in his proof which was repaired in [7].
As proved by Pugh and Robinson, the $C^1$ Closing Lemma states that if
$p$ is a nonwandering point of a $C^1$ vector field $X$ on a compact manifold
$M$ then every neighborhood of $X$ in the $C^1$ topology contains a vector
field $Y$ having a periodic orbit through $p$. The $C^r$ Closing Lemma says
that if $X$ is $C^r$ then $Y$ can be found in any $C^r$ neighborhood of $X$,
$r \geq 0$.

For $r > 1$ the $C^r$ Closing Lemma has not yet to be verified, even
generically, and is known only for very special cases. For detailed his-
torical comments, see [2, 4, 5, 6, 7].

Peixoto and Pugh [4] improved the above results at the chain re-
current points, which are the weakest type of recurrence in dynamics
theory. To be precise, they proved that any chain recurrent point of a
$C^r$ vector field $X$ on the plane $\mathbb{R}^2$ can be periodic under a $C^r$ pertur-
bation of $X$ in the $C^r$ Whitney topology if every fixed point of $X$ is
hyperbolic.

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Until 1992, there was an open fundamental question in dynamics, reasonably called the $C^1$ Connecting Lemma:

for a flow $\phi$ on a manifold $M$ and $p, q \in M$, we suppose

$$\omega(p) \cap \alpha(q) \neq \emptyset$$

where $\alpha(q) = \{ x \in M : \phi_{t_n}(q) \to x \text{ for some } t_n \to -\infty \}$. does there exist a flow that $C^1$-approximates $\phi$ for which $p, q$ lie on the same orbit?

Pugh gave an example to show that the $C^1$ Connecting Lemma is false. The example was constructed in the plane using the concept of the flow plug. Also he constructed a $C^1$ flow $\phi$ on the punctured torus, $T^2 - \{0\}$, with chain recurrent orbits which cannot be periodic using $C^1$ small perturbations (see figure 8 in [6]). Consequently, he disproves the $C^1$-Closing Lemma for chain recurrence on noncompact 2-manifolds in general, but on compact 2-manifolds it is still an open question (for more details, see [6]).

The purpose of this note is to try to solve the above open question, using the technics which are used to prove the $C^0$ Closing Lemma for chain recurrence in noncompact n-manifolds.

Throughout the paper, we will follow the definitions and notations given in [2].

**Theorem.** Let $M$ be a compact orientible $C^\infty$-manifold of dimension 2, $X$ a $C^r$ vector field on $M$ with $r \geq 1$, and $p$ a chain recurrent point of $X$. For each neighborhood $U$ of $X$ in the space $\mathcal{X}^r(M)$ of $C^r$ vector fields on $M$ with the $C^r$ topology, there exists $Y \in U$ such that $Y$ has a closed orbit through $p$.

**Proof.** Let $p \in M$ be a nontrivial chain recurrent point for $X$, and $\varepsilon > 0$ be arbitrary. Using the same technics as in the proof of Lemma 3.5 in [2], we can choose $0 < a < \delta(p)$ such that for any time $t \in [-a, a]$, two points $O(\phi^+(p, t)) \cap O^+(p, -\delta(p))$ and $p$ can be connected by a trajectory arc of a vector field $Y \in \mathcal{U}(X, \varepsilon)$, with $Y = X$ outside $N_p$. Choose a continuous function $\xi : M_0 \to (0, \infty)$ such that

1. $B(p, \xi(p)) \subset \{ O(\phi^+(p, s)) \cap O^+(\phi(p, t)) : |t| \leq \delta(p), |s| \leq \frac{\delta}{2} \}$
2. $B(x, \xi(x)) \subset \{ O(\phi^+(x, s)) \cap O^+(\phi(x, t)) : |t| \leq \delta(x), |s| \leq \frac{1}{3} c(x) \delta(x) \}$, if $x \neq p$. 

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Since $p$ is chain recurrent, there exists $(\xi, 1)$-chain $(p_1, t_1), \ldots, (p_n, t_n)$ from $p$ to $p$. Since $d(\phi(p_i, t_i), p_{i+1}) < \xi(p_{i+1})$ for $1 \leq i \leq n - 1$, we can choose a $C^{r+1}$-curve $\alpha_i : [0, b_i] \to M_0$ such that

1. $\alpha_i(0) = x_i = [p_i, \phi(p_i, t_i)] \cap O^\perp(\phi(p_{i+1}, -\delta(p_{i+1})))$
2. $\alpha_i(b_i) = y_{i+1} = \phi(p_{i+1}, \delta(p_{i+1}))$
3. $\dot{\alpha}_i(0) = X_{x_i}, \dot{\alpha}_i(b_i) = X_{y_{i+1}}$
4. $\|\dot{\alpha}_i - X\|_r < \varepsilon$

Similarly we can choose a $C^{r+1}$-curve $\alpha_n : [0, b_n] \to M_0$ such that

1. $\alpha_n(0) = x_n = [p_n, \phi(p_n, t_n)] \cap O^\perp(\phi(p, -\delta(p))$
2. $\alpha_n(b_n) = p, \dot{\alpha}_n(0) = X_{x_n}, \dot{\alpha}_n(b_n) = X_p$
3. $\|\dot{\alpha}_n - X\|_r < \varepsilon$

In this way, we can construct a simple closed $C^{r+1}$-curve $\alpha$ in $M_0$ such that

1. $\alpha(0) = p = \alpha(T), \alpha(t + T) = \alpha(t)$, for some $T > 0$
2. $\|\dot{\alpha} - X\|_r < \varepsilon$

Even if the flow boxes are overlap, we can select a periodic curve $\alpha$ which we want. Since $X^\perp_x \neq 0$ for any $x \in \alpha$, there exists a neighborhood $U$ of $\alpha$ in $M_0$ such that $X^\perp_p \neq 0$ for any $p \in \bar{U}$. Choose $b > 0$ such that $\phi^\perp(\alpha \times [-b, b]) \subset U$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-bump function such that $f(t) = 0$ for $|t| \geq b$, $f(0) = 1$, and $0 < f(t) < 1$ for $0 < |t| < b$. Define a vector field $Y$ on $M$ as follow; for any $x \in M$,

$$Y(x) = \begin{cases} X_x + f(t)V_y(t), & \text{if } x = \phi^\perp(y, t), y \in \alpha, |t| \leq b \\ X_x, & \text{if } x \notin \phi^\perp(\alpha \times [-b, b]) \end{cases}$$

where $V_y(t)$ is the parallel transport of $\dot{\alpha}(y) - X_y$ along $O^\perp(y)$. Then we have $\|X - Y\|_r < \varepsilon$ and $p$ is a periodic point of $Y$. This completes the proof. \qed

References

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