ON DERIVATIONS IN NONCOMMUTATIVE SEMISIMPLE BANACH ALGEBRAS

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ABSTRACT. The purpose of this paper is to prove the following results: Let \( A \) be a noncommutative semisimple Banach algebra. (1) Suppose that a linear derivation \( D : A \rightarrow A \) is such that \( \langle D(x), x \rangle x = 0 \) holds for all \( x \in A \). Then we have \( D = 0 \). (2) Suppose that a linear derivation \( D : A \rightarrow A \) is such that \( D(x)x^2 + x^2D(x) = 0 \) holds for all \( x \in A \). Then we have \( D = 0 \).

1. Introduction

Throughout this paper \( R \) will represent an associative ring with center \( Z(R) \), and \( A \) will represent an algebra over a complex field. The commutator \( xy - yx \) will be denoted by \( [x, y] \). We make use of the basic commutator identities \( [xy, z] = [x, z]y + x[y, z], [x, yz] = [x, y]z + y[x, z] \). An additive mapping \( D \) from \( R \) to \( R \) is called a derivation if \( D(xy) = D(x)y + xD(y) \) holds for all \( x, y \in R \). A derivation \( D \) is inner if there exists \( a \in R \) such that \( D(x) = [a, x] \) holds for all \( x \in R \). Recall that a ring \( R \) is prime if \( aRb = (0) \) implies that either \( a = 0 \) or \( b = 0 \). Sinclair [1] has proved that every linear derivation on a semisimple Banach algebra is continuous. Singer and Wermer [4] state that every continuous linear derivation on a commutative Banach algebra maps the algebra into its Jacobson radical. Combining these two results we obtain that there are no nonzero derivations on a commutative semisimple Banach algebra. Now it seems natural to ask, under what additional assumptions a linear derivation on a noncommutative semisimple Banach algebra is zero. It is our aim in this paper to give answers to the question above.

Received September 26, 1997.
1991 Mathematics Subject Classification: 46H05, 46J05.
Key words and phrases: derivation, inner derivation, prime ring, semisimple Banach algebra.
2. The results

We now state and prove the main results.

**Theorem 2.1.** Let $A$ be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $D : A \to A$ such that $[D(x), x]x = 0$ holds for all $x \in A$. Then we have $D = 0$.

*Proof of Theorem 2.1.* For the proof of the theorem we shall need the following purely algebraic result which can be proved without any specific knowledge concerning prime rings.

**Lemma 2.1.** Let $R$ be a noncommutative prime ring of characteristic different from two and $I$ a nonzero two-sided ideal of $R$. Suppose that there exists a derivation $D : R \to R$ such that $[D(x), x]x = 0$ holds for all $x \in I$. Then we have $D = 0$ on $R$.

*Proof of Lemma 2.1.* We define a mapping $B(., .) : I \times I \to I$ by the relation

\[
(1) \quad B(x, y) = [D(x), y] + [D(y), x], \quad x, y \in I.
\]

Obviously, $B(x, y) = B(y, x)$ for all $x, y \in I$ and $B(., .)$ is additive in both arguments. Moreover, a simple calculation shows that the relation

\[
(2) \quad B(xy, z) = B(x, z)y + xB(y, z) + D(x)[y, z] + [x, z]D(y)
\]

holds for all $x, y, z \in I$. We shall write $f(x)$ for $B(x, x)$. Then

\[
(3) \quad f(x) = 2[D(x), x], \quad x \in I.
\]

It is easy to see that

\[
(4) \quad f(x + y) = f(x) + f(y) + 2B(x, y)
\]

is fulfilled for all $x, y \in I$. Now the assumption of the lemma can be written as follows

\[
(5) \quad f(x)x = 0, \quad x \in I.
\]
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The linearization of (5) gives

\[ 0 = (f(x) + f(y) + 2B(x, y))(x + y) \]
\[ = f(x)x + f(y)x + 2B(x, y)x + f(x)y + f(y)y + 2B(x, y)y, \]

which reduces to

\[ f(x)y + f(y)x + 2B(x, y)x + 2B(x, y)y = 0, \quad x, y \in I. \]

Replacing \( x \) by \( -x \) in (7), and subtracting the new result from (7), we have

\[ f(x)y + 2B(x, y)x = 0, \quad x, y \in I, \]

since \( R \) is of characteristic not two. Let \( y \) be \( yx \) in (8). Then, by (5) and (8), we get

\[ 0 = f(x)yx + 2B(x, yx)x \]
\[ = 2[y, x]D(x)x, \quad x, y \in I. \]

Hence we arrive at

\[ [y, x]D(x)x = 0, \quad x, y \in I, \]

since \( R \) is of characteristic not two. We intend to prove that

\[ D(x)x = 0 \]

holds for all \( x \in I \). Suppose on the contrary that \( D(a)a \neq 0 \) for some \( a \in I \). Note that \( I \) is a non-zero noncommutative prime ring of characteristic not two and a mapping \( y \mapsto [y, a] \) is an inner derivation on \( I \). Then (10)and Lemma 1 in [2] imply \( a \in Z(I) \). We have therefore proved that \( D(x)x = 0 \) in case \( x \notin Z(I) \). It remains to prove that \( D(x)x = 0 \) also in the case when \( x \in Z(I) \). Let \( x \in Z(I) \) and let \( y \notin Z(I) \). We have also \( x + y \notin Z(I) \). We see that \( D(y)y = 0 \) and
\[ D(x + y)(x + y) = 0. \] Then \[ 0 = (D(x) + D(y))(x + y) = D(x)x + D(x)y + D(y)x + D(y)y = D(x)x + D(x)y + D(y)x. \] Hence

(12) \[ D(x)x + D(x)y + D(y)x = 0. \]

Replacing \( x \) by \(-x\) in (12), we have

(13) \[ D(x)x - D(x)y - D(y)x = 0. \]

From (12) and (13) it follows \( D(x)x = 0 \), which completes the proof of (11). The linearization of (11) leads to

(14) \[ D(x)y + D(y)x = 0, \ x, y \in I. \]

Substituting \( zy \) for \( y \) in (14), we get

(15) \[ 0 = D(x)zy + D(zy)x \\
= D(x)zy + zD(y)x + D(z)y, \ x, y \in I, \ z \in R. \]

Combining (14) with (15), we obtain

(16) \[ [D(x), z]y + D(z)y = 0, \ x, y \in I, \ z \in R. \]

Replacing \( z \) by \( D(x) \) in (16), we have

\[ D^2(x)yx = 0, \ x, y \in I. \]

Since \( I \) is prime, we know that \( D^2(x) = 0 \) holds for all \( x \in I \). This yields \( D(x) = 0 \) for all \( x \in I \) by Theorem 1 in [2]. Now, substituting \( rx \) (\( r \in R \)) for \( x \), we have \( D(r)x = 0 \), that is, \( D(r)I = 0 \). Since \( R \) is prime and \( I \) is nonzero, it follows that \( D(r) = 0 \) for all \( r \in R \). The proof of Lemma 2.1 is complete. \( \square \)
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Proof of Theorem 2.1 continued. By the result of Johnson and Sinclair [1] every linear derivation on a semisimple Banach algebra is continuous. Sinclair [3] has proved that every continuous linear derivation on a Banach algebra leaves the primitive ideals of $A$ invariant. Hence for any primitive ideal $P \subset A$, we can define a linear derivation $D_P : A/P \to A/P$ by $D_P(x + P) = D(x) + P$, $x \in A$. The assumption of the theorem $[D(x), x]x = 0, x \in A$ gives $[D_P(x + P), x + P](x + P) = P, x \in A$. Since $P$ is a primitive ideal, $A/P$ is prime. Hence, in case $A/P$ is noncommutative, we have $D_P = 0$, since all the assumptions of Lemma 2.1 are fulfilled. In case $A/P$ is commutative, we can conclude that $D_P = 0$ as well since $A/P$ is semisimple and since we know that there are no nonzero linear derivations on commutative semisimple Banach algebras. This implies that $D(x)$ is in the intersection of all primitive ideals of $A$ for all $x \in A$. Since the intersection of all primitive ideals is the Jacobson radical, and $A$ is semisimple, it follows $D = 0$. The proof of Theorem 2.1 is complete.

As a special case of Theorem 2.1 we obtain the following result which characterizes commutative semisimple Banach algebras among all semisimple Banach algebras.

\[ \Box \]

Corollary 2.1. Let $A$ be a semisimple Banach algebra. Suppose that $[[x, y], x]x = 0$ holds for all $x, y \in A$. Then $A$ is commutative.

Theorem 2.2. Let $A$ be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $D : A \to A$ such that $D(x)x^2 + x^2D(x) = 0$ holds for all $x \in A$. Then we have $D = 0$.

Proof of Theorem 2.2. For the proof of Theorem 2.2 as in Theorem 2.1 we also need prove the following algebraic result.

\[ \Box \]

Lemma 2.2. Let $R$ be a noncommutative prime ring of characteristic different from two and $I$ a nonzero two-sided ideal of $R$. Suppose that there exists a derivation $D : R \to R$ such that $D(x)x^2 + x^2D(x) = 0$ holds for all $x \in I$. Then we have $D = 0$ on $R$.

Proof of Lemma 2.2. Suppose that

\[ D(x)x^2 + x^2D(x) = 0 \]
holds for all $x \in I$. The linearization of (1) leads to

$$0 = D(x)xy + D(x)yx + D(x)y^2 + D(y)x^2 + D(y)xy$$

$$+ D(y)yx + x^2D(y) + xyD(x) + xyD(y)$$

$$+ yxD(x) + yxD(y) + y^2D(x), \ x, y \in I. \quad (2)$$

Replacing $y$ by $-y$ in (2), and subtracting the result from (2), we have

$$D(x)xy + D(x)yx + D(y)x^2 + x^2D(y)$$

$$+ xyD(x) + yxD(x) = 0, \ x, y \in I, \quad (3)$$

since $R$ is of characteristic not two. Substituting $xy$ for $y$ in (3) and using (1), we arrive at

$$D(x)xyx + xD(x)x^2 + D(x)yx + x^3D(y)$$

$$+ x^2yD(x) + xxyD(x) = 0, \ x, y \in I. \quad (4)$$

Left multiplication of (3) by $x$ leads to

$$xD(x)xy + xD(x)yx + xD(y)x^2 + x^3D(y)$$

$$+ x^2yD(x) + xyxD(x) = 0, \ x, y \in I. \quad (5)$$

Subtracting (5) from (4), we obtain

$$-xD(x)xy + [D(x), x]yx + D(x)y^2 = 0, \ x, y \in I. \quad (6)$$

Replacing $y$ by $yD(x)$ in (6), we have

$$-xD(x)xyD(x) + [D(x), x]yD(x)x + D(x)yD(x)x^2 = 0, \ x, y \in I. \quad (7)$$

Right multiplication of (6) by $D(x)$ gives

$$-xD(x)xyD(x) + [D(x), x]yxD(x) + D(x)y^2D(x) = 0, \ x, y \in I. \quad (8)$$

Subtracting (8) from (7), we obtain

$$D(x)y[D(x), x^2] + [D(x), x]y[D(x), x] = 0, \ x, y \in I. \quad (9)$$
Putting $xy$ instead of $y$ in (9), it follows that

$$D(x)xy[D(x), x^2] + [D(x), x]xy[D(x), x] = 0, \ x, y \in I. \ (10)$$

Left multiplication of (10) by $x$ gives

$$xD(x)y[D(x), x^2] + x[D(x), x]y[D(x), x] = 0, \ x, y \in I. \ (11)$$

Subtracting (11) from (10), we obtain

$$[D(x), x]y[D(x), x^2] + [[D(x), x], x]y[D(x), x] = 0, \ x, y \in I. \ (12)$$

Replacing $y$ by $y[D(x), x]z$ in (12), we get

$$[D(x), x]y[D(x), x]z[D(x), x^2] + [[D(x), x], x]y[D(x), x]z[D(x), x] = 0, \ x, y, z \in I.$$

Using (12) we can write $-[[D(x), x], x]z[D(x), x]$ for $[D(x), x]z[D(x), x^2]$ and $-[D(x), x]y[D(x), x^2]$ for $[[D(x), x], x]y[D(x), x]$ in the above calculation.

Hence we arrive at

$$[D(x), x]y[[D(x), x], x]z[D(x), x] + [D(x), x]y[D(x), x^2]z[D(x), x] = 0,$$

which can be reduced in the form

$$[D(x), x]y[D(x), x]xz[D(x), x] = 0, \ x, y, z \in I.$$

Since $I$ is a prime ring, it follows that

$$[D(x), x]x = 0$$

holds for all $x \in I$. This implies that $D = 0$ on $R$ by Lemma 2.1. The proof of Lemma 2.2 is complete. \[\square\]
Proof of Theorem 2.2 continued. Let us use the same argument as that used to prove Theorem 2.1. Then we can define a linear derivation $D_P : A/P \to A/P$, where $P$ is any primitive ideal of $A$, by $D_P(x + P) = D(x) + P$, $x \in A$. Also the assumption of Theorem 2.2 $D(x)x^2 + x^2D(x) = 0$, $x \in A$ gives $D_P(x + P)(x + P)^2 + (x + P)^2D_P(x + P) = P$, $x \in A$. Hence Lemma 2.2 deduces $D_P = 0$, and semisimplicity of $A$ forces $D = 0$. The proof of Theorem 2.2 is complete.

We also obtain the following result as a special case of Theorem 2.2 as in Corollary 2.1.

Corollary 2.2. Let $A$ be a semisimple Banach algebra. Suppose that $[x, y]x^2 + x^2[y, x] = 0$ holds for all $x, y \in A$. Then $A$ is commutative.

References


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