A CHARACTERIZATION OF SPACE FORMS

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ABSTRACT. For a Riemannian manifold \((M^n, g)\), we consider the space \(V(M^n, g)\) of all smooth functions on \(M^n\) whose Hessian is proportional to the metric tensor \(g\). It is well-known that if \(M^n\) is a space form then \(V(M^n)\) is of dimension \(n + 2\). In this paper, conversely, we prove that if \(V(M^n)\) is of dimension \(\geq n + 1\), then \(M^n\) is a Riemannian space form.

1. Introduction

Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold \((n \geq 2)\) with the Riemannian connection \(\nabla\). For \(f \in C^\infty(M^n)\), the Hessian \(H^f\) of \(f\) is a symmetric \((0,2)\) tensor field on \(M^n\) defined by

\[
H^f(X, Y) = g(\nabla_X \nabla f, Y), \quad X, Y \in TM,
\]

where \(\nabla f\) denotes the gradient vector field of \(f\). Let \(V(M^n)\) be the space of all smooth functions on \(M^n\) whose Hessian is proportional to the metric tensor \(g\) and \(m(V^n)\) denote the dimension of \(V(M^n)\). Then clearly we have \(m(M^n) \geq 1\) for any \((M^n, g)\). For the Riemannian space forms \(M^n = S^n(r), H^n(r)\) or \(E^n\), we have \(m(M^n) = n + 2\), respectively (see §2).

Hence it is natural to ask the following question ([5,10,14]):

To what extent does \(m(M^n)\) determine the geometrical and topological structure of \((M^n, g)\)?

And they proved, in our terminology, the following ([10,14]):

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THEOREM. Let \((M^n, g)\) be a complete connected Riemannian manifold with \(m(M^n) \geq 2\). Then the number \(N\) of critical points of a non constant \(f \in V(M^n)\) is less than or equal to 2 and \(M^n\) is conformally diffeomorphic to

(i) the Euclidean sphere \(S^n(N = 2)\),
(ii) the Euclidean space \(E^n\), or the hyperbolic space \(H^n(N = 1)\),
(iii) the Riemannian product \(I \times F\), where \((F, g_F)\) is a complete \((n-1)\)-dimensional Riemannian manifold and \(I\) is an open interval \((N = 0)\).

In this article, we study the manifolds \(M^n\) with \(m(M^n) \geq 3\). As a result, we prove the following:

THEOREM A. Let \((M^n, g)\) be a compact connected Riemannian manifold. If \(m(M^n) \geq 3\), then \(M^n\) is isometric to the Euclidean sphere \(S^n(r)\).

THEOREM B. Let \((M^n, g)\) be a complete noncompact connected manifold. If \(m(M^n) \geq n + 1\), then \(M^n\) is isometric to the Euclidean space \(E^n\) or the hyperbolic space \(H^n(r)\).

These results are sharp in the sense that (1) the ellipsoid of revolution \(M^n\) in \(R^{n+1}\) defined by \(a^2 x_1^2 + b^2 (x_2^2 + \cdots + x_{n+1}^2) = 1, \ a \neq b,\) has \(m(M^n) = 2\), (2) if \(M^n\) is the cylinder \(R^{n-1} \times S^1\), then we have \(m(M^n) = n\).

It is obvious that for 1-dimensional manifold \(M^1\), we have \(m(M^1) = \infty\).

2. Examples

EXAMPLE 1. Euclidean space \(E^n\)

It is straightforward to show that

\[
V(E^n) = \left\{ a \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} a_i x_i + b|a, a_1, \cdots, a_n, b \in R \right\},
\]

where \((x_1, \cdots, x_n)\) is the rectangular coordinates for \(E^n\).
EXAMPLE 2. Euclidean sphere $S^n(r)$

For the Euclidean sphere $S^n(r) = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = r^2\}$, it is not hard to show that the restriction $f$ of each function of the form: $\sum_{i=1}^{n+1} a_i x_i + b$, $a_i, b \in \mathbb{R}$, belongs to $V(S^n(r))$. Conversely, each nonconstant function $f \in V(S^n(r))$ satisfies $\Delta f + (n/r^2)f = c$ (constant) (see Lemma 3.1). Hence $h = f - cr^2/n$ is a nonconstant eigenfunction of $S^n(r)$ with eigenvalue $n/r^2$. Therefore we see that $h$ is a first eigenfunction of $S^n(r)([6])$, that is, $f$ is the restriction of one of the above functions.

EXAMPLE 3. Warped product space $I \times_w F^{n-1}$ with dim I=1.

For an (n-1)-dimensional Riemannian manifold $(F, g_F)$ let $M^n$ be the warped product space $I \times_w F$ with metric $g = dt^2 + w(t)^2 g_F$, where $I$ is an open interval and $w(t)$ is a positive function on $I$ ([4,13]). Then for the function $f$ defined by $f(t) = a \int_{t_0}^t w(t)dt + b$, $a, b \in \mathbb{R}$, it can be shown that $H^f$ is proportional to $g$. Hence we have $m(M^n) \geq 2$.

EXAMPLE 4. Hyperbolic space $H^n(r)$

Let $R^{n+1}_1$ be the Lorentz-Minkowski space with metric tensor $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$. Then we have

$$H^n(r) = \{x \in R^{n+1}_1 | \langle x, x \rangle = -r^2, \quad x_{n+1} > 0\}.$$ 

It is straightforward, as in Example 2, to show that the restriction $f$ of each function of the form: $\sum_{i=1}^{n+1} a_i x_i + b$, $a_i, b \in \mathbb{R}$, belongs to $V(H^n(r))$.

Conversely, using the geodesic polar coordinates centered at $(0, \cdots, 0, r)$ or equivalently, the warped product structure $H^n(r) = [0, \infty) \times_w S^{n-1}(1)$ with $w(t) = r \sinh t$, it is not hard to show that each function $f \in V(H^n(r))$ is the restriction of one of the above functions.

3. Basic formulas

Let $(M^n, g)$ be a Riemannian manifold of n-dimensional, $\nabla$ the Riemannian connection and

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

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the curvature tensor. We write also $\langle X, Y \rangle$ instead of $g(X, Y)$ if this is convenient.

Note that for $f \in C^\infty(M^n)$, $f$ belongs to $V(M^n)$ if and only if

\begin{equation}
\nabla_X\nabla f = \varphi X, \quad X \in TM, \quad \text{where} \quad \varphi = \Delta f / n.
\end{equation}

Then it is easy to prove the following:

**Lemma 3.1.** If $f \in V(M^n)$, then we have for $X, Y \in TM$
\begin{enumerate}
\item $(i) \quad R(X,Y)\nabla f = X(\varphi)Y - Y(\varphi)X$
\item $(ii) \quad Ric(X, \nabla f) = -(n-1)\langle X, \nabla \varphi \rangle, \quad \text{where} \quad \varphi = \Delta f / n.$
\end{enumerate}

Thus for an Einstein manifold $M^n$ with $Ric = (n-1)Kg$ we see that $\Delta f + nKf$ is constant.

**Lemma 3.2.** Let $U$ be an open set without critical points of a function $f \in V(M^n)$ and let $e_1$ be the unit vector field in the direction of $\nabla f$. Then we have the following:
\begin{enumerate}
\item $(i)$ the integral curve $\gamma(t)$ of $e_1$ is a geodesic and the level hypersurfaces of $f$ are totally umbilic.
\item $(ii)$ $w = |\nabla f|$ and $n\varphi = \Delta f$ are constant along the levels of $f$.
\item $(iii)$ For any vector field $X$, the sectional curvature $K = K(X \wedge \nabla f)$ of the section spanned by $X$ and $\nabla f$ is constant along the levels of $f$ and does not depend on $X$.
\item $(iv)$ Along $\gamma(t)$, $w(t) = w(\gamma(t))$ and $K(t) = K(\gamma(t))$ are related by
\begin{equation}
w''(t) + K(t)w(t) = 0.
\end{equation}
\end{enumerate}

**Proof.** The equation (3.1) reads as follows:

\[ X(w)e_1 + w\nabla_X e_1 = \varphi X \quad \text{for all} \quad X \in TM, \]

or equivalently

\begin{equation}
e_1(w) = \varphi, \quad \nabla_{e_1} e_1 = 0 \quad \text{and}
\end{equation}

\begin{equation}
\nabla_X e_1 = \frac{\varphi}{w} X, \quad X(w) = 0 \quad \text{for all} \quad X \perp e_1.
\end{equation}
Since we have
\[(3.5)\]
\[X\varphi = X(e_1w) = e_1(Xw) + [X, e_1](w) = (\nabla_X e_1)(w) - (\nabla_{e_1} X)(w),\]
(3.3) and (3.4) imply that \(X\varphi = 0\) for all \(X \perp e_1\). For any unit vector \(X\) such that \(X \perp e_1\), we have from Lemma 3.1
\[
K(X \wedge e_1) = \langle R(X, e_1)e_1, X \rangle \\
= \frac{1}{w}[(X(\varphi)e_1 - e_1(\varphi)X, X)] \\
= -\frac{e_1(\varphi)}{w},
\]
and as in (3.5) we may prove that \(X(e_1\varphi) = 0\) for all \(X \perp e_1\). This completes the proof. □

For a fixed point \(p \in U\), choose an orthonormal basis \(e_2, \cdots, e_n\) of the level hypersurface \(F\) through \(p\). We denote the parallel translates along \(\gamma(t)\) of \(e_2, \cdots, e_n\) by the same notation. For each \(i \in \{2, \cdots, n\}\) let \(\eta_i(s)\) be the integral curve of \(e_i\) in \(F\) with \(\eta_i(0) = p\), and \(x(s, t)\) be the one parameter family \(\exp_{\eta_i(s)}(te_1)\) of geodesics. Then \(J_i(t) = x_s(0, t)\) is a Jacobi vector field along \(\gamma\), so that it satisfies the Jacobi equation
\[(3.6)\]
\[J_{ii}(t) + R(J_i, e_1)e_1 = 0 \quad \text{with} \quad J_i(0) = e_i, \quad J_i'(0) = \frac{w'(0)}{w(0)} e_i,
\]
where \(w'(t)\) denotes the derivative with respect to \(t\).

Since \(e_i\) is parallel along \(\gamma\), (3.2) and the Jacobi equation show that \(\langle J_i(t), e_1 \rangle = 0\) and
\[(3.7)\]
\[\langle J_i(t), e_j \rangle'' + K(t)\langle J_i(t), e_j \rangle = 0, \quad j \in \{2, \cdots, n\}.
\]
Hence from (3.2) and the initial condition we obtain
\[(3.8)\]
\[J_i(t) = \frac{w(t)}{w(0)} e_i.
\]
This shows that the metric \(g\) of \(M\) is locally given by \(g = dt^2 + \frac{w(t)^2}{w(0)^2}g_F\), that is, \(M\) is locally the warped product space \(I \times_{w(t)/w(0)} F\).

Thus from Example 3 of §2, we obtain the following ([10,14]):
LEMMA 3.3. The following conditions are equivalent:
(i) There exists a neighborhood $U$ of $p$ and a function $f \in V(U)$ with $\nabla f(p) \neq 0$.
(ii) There exists a neighborhood $U$ of $p$ such that $U$ is a warped product space $I \times_w F$ with 1-dimensional base $I$.

In his Thesis ([7]), K. L. Easley found an equivalent condition of the above lemma (see also [1]).

The situation is quite different near a critical point of $f$. W. Kühnel proved that the critical points of a nonconstant function $f \in V(M^n)$ are isolated ([10, 14]). Around a critical point $p \in M$ of a nonconstant function $f \in V(M)$ we may prove the following ([10]):

LEMMA 3.4. The following conditions are equivalent:
(i) There exists a neighborhood $U$ of $p$ and a nonconstant function $f \in V(U)$ with $\nabla f(p) = 0$.
(ii) There exists polar coordinates $(t, u_1, \ldots, u_{n-1})$ in a neighborhood of $p$ and an even function $f = f(t)$ with $f'(0) = 0$ and $f''(0) \neq 0$ such that

$$g = dt^2 + \frac{f'(t)^2}{f''(0)^2} g_1,$$

where $g_1$ denotes the metric of the Euclidean unit sphere $S^{n-1}(1)$.

(iii) There exists a function $\lambda(t)$ defined on $(0, t_0)$ such that, for each $\xi \in T_p M$, $|\xi| = 1$, and each $t \in (0, t_0)$ the shape operator $S_p(m)$ of the geodesic sphere $G(p, t)$ at $m = \exp_p(t\xi)$ satisfies $S_p(m) = \lambda(t)I$.

Proof. Equivalence of (i) and (ii) was given in [10]. (i) $\Rightarrow$ (iii) is given by (3.4) with $\lambda(t) = w'(t)/w(t)$. Suppose that (iii) holds. Then Theorem 12 in [9] implies that, with respect to any normal coordinates centered at $p$, the metric tensor $g$ is given by the formula:

$$g = h^2(t) \sum_{i=1}^{n} (dx^i)^2 + \frac{1 - h^2(t)}{t^2} (\sum_{i=1}^{n} x^i dx^i)^2,$$

where $h(t) = \exp\{\int_0^t (\lambda(r) - \frac{1}{r}) dr\}$, and $t^2 = \sum_{i=1}^{n} x_i^2$.

Since $\sum_{i=1}^{n} (dx^i)^2 = dt^2 + t^2 ds_1^2$, where $ds_1^2$ is the line element of the unit sphere in $T_p M$, we get by substitution

$$g = dt^2 + w^2(t) ds_1^2, \quad w(t) = th(t).$$
Now we may prove that \( f(t) = \int_0^t w(r)dr \) satisfies (3.1) on the geodesic ball of radius \( t_0 \) around \( p \). \( \square \)

Note that for a critical point \( p \) of \( f \in \mathcal{V}(M^n) \) the level hypersurfaces of \( f \) coincide with the geodesic spheres around \( p \) and \( p \) is an isotropic point with curvature \( K(0) \). For the space form \( E^n, S^n(r) \) or \( H^n(r) \) we have

\[
\lambda(t) = \frac{1}{t}, \frac{1}{r} \cot \frac{t}{r} \quad \text{or} \quad \frac{1}{r} \coth \frac{t}{r},
\]

respectively.

4. Proofs

In this section we prove the main theorems and we assume that \( M^n \) is connected and complete.

First, we give some lemmas. Note that for any nonconstant function \( f \in \mathcal{V}(M^n) \), the number \( N \) of critical points of \( f \) is less than or equal to 2 and \( M^n \) is conformally diffeomorphic to \( S^n(N = 2) \), \( E^n \) or \( H^n(N = 1) \) and \( I \times F(N = 0) \) ([10]).

**Lemma 4.1.** Let \( f_1, f_2 \in \mathcal{V}(M^n) \) with \( \nabla f_1(p) = \nabla f_2(p) = 0 \). Then \( \{f_1, f_2, 1\} \) is linearly dependent on \( M^n \).

**Proof.** We may assume that \( f_1, f_2 \) are nonconstant functions. For any radial geodesic \( \gamma(t) \) emanating from \( p \), let \( w_i(t) = |\nabla f_i|((\gamma(t))) \), \( i = 1, 2 \). Then \( w_i(t) \) satisfies

\[
(3.2) \quad w_i''(t) + K(t)w_i(t) = 0 \quad \text{with} \quad w_1(0) = w_2(0) = 0.
\]

Hence we have \( w_2(t) = aw_1(t) \), where \( a = w_2(0)/w_1(0) \). Since \( f_i(t) = \int_0^t w_i(t)dt + f_i(0) \), this completes the proof. \( \square \)

**Lemma 4.2.** Let \( f_1, f_2, \ldots, f_k \in \mathcal{V}(M^n) \) with \( k \leq n \). If \( \dim(\{\nabla f_1(p), \ldots, \nabla f_k(p)\}) \leq k - 1 \) at each point \( p \) in an open set \( U \), then \( \{f_1, \ldots, f_k, 1\} \) is linearly dependent on \( M^n \).
Proof. For \( k = 1 \), suppose that \( \nabla f_1 \equiv 0 \) on an open set \( U \). Then \( f_1 \) must be constant, because \( f_1 \) has infinitely many critical points.

Now assume that the lemma holds for \( k \leq n - 1 \). Suppose that \( \dim(\{\nabla f_1(p), \cdots, \nabla f_{k+1}(p)\}) \leq k \) on \( U \) with \( k+1 \leq n \). Then by induction hypothesis we may assume that \( U_1 = \{p \in U | \dim(\{\nabla f_1(p), \cdots, \nabla f_k(p)\}) = k\} \) is a nonempty open set. On \( U_1 \) we have \( \nabla f_{k+1} = h_1 \nabla f_1 + \cdots + h_k \nabla f_k \). And (3.1) shows that for all \( X \) we have

\[
(4.1) \quad \varphi_{k+1} X = X(h_1) \nabla f_1 + \cdots + X(h_k) \nabla f_k + (h_1 \varphi_1 + \cdots + h_k \varphi_k) X,
\]

where \( \varphi_i = \Delta f_i / n, i = 1, \cdots, k + 1 \). Since \( k \leq n - 1 \) we can choose \( X \) so that \( X \) is orthogonal to \( \{\nabla f_1, \cdots, \nabla f_k\} \). Hence (4.1) implies that

\[
(4.2) \quad \varphi_{k+1} = h_1 \varphi_1 + \cdots + h_k \varphi_k.
\]

By (4.2) together with (4.1) we see that \( h_1, \cdots, h_k \) are constants \( a_1, \cdots, a_k \), respectively. Therefore the gradient of \( f_{k+1} - (a_1 f_1 + \cdots + a_k f_k) \in V(M^n) \) vanishes on \( U_1 \), in particular, it has infinitely many critical points. Thus it must be constant on \( M^n \). This completes the proof.

\[\square\]

Lemma 4.3. Let \( c \) be a regular value of \( f_1 \in V(M^n) \) and \( F_1 \) be the hypersurface \( f_1^{-1}(c) \). Then

(i) For any \( f \in V(M^n) \), the restriction \( \tilde{f} = f|_{F_1} \) of \( f \) belongs to \( V(F_1) \).

(ii) If \( \{f_1, \cdots, f_k, 1\} \) is a linearly independent subset of \( V(M^n) \), then \( \{\tilde{f}_2, \cdots, \tilde{f}_k, 1\} \) is linearly independent. Hence we have \( m(F_1) \geq m(M^n) - 1 \).

Proof. (i) Let \( \tilde{\nabla} \) be the induced connection of \( F_1 \) and \( \tilde{\nabla} \tilde{f} \) be the gradient vector field of \( \tilde{f} \) on \( F_1 \). Then, using Lemma 3.2, a direct computation shows that for all \( X \in TF_1 \)

\[
\tilde{\nabla}_X \tilde{\nabla} \tilde{f} = \left( \varphi - \frac{\varphi_1 \langle \nabla f, \nabla f_1 \rangle}{\langle \nabla f_1, \nabla f_1 \rangle} \right) X,
\]

where \( \varphi = \Delta f / n \) and \( \varphi_1 = \Delta f_1 / n \).
(ii) Suppose that $a_2 \tilde{f}_2 + \cdots + a_k \tilde{f}_k + b = 0$ on $F_1$. Then we have on $F_1, \nabla (a_2 f_2 + \cdots + a_k f_k) = h \nabla f_1$ for some function $h$ on $F_1$. By (3.1) we get for all $X \in TF_1, (a_2 \varphi_2 + \cdots + a_k \varphi_k)X = X(h) \nabla f_1 + h \varphi_1 X$. Thus we see that $h$ is a constant $c$, which implies that $cf_1 - (a_2 f_2 + \cdots + a_k f_k)$ is a function in $V(M^n)$ which has infinitely many critical points. Therefore $cf_1 - (a_2 f_2 + \cdots + a_k f_k)$ must be constant. This completes the proof. □

Proof of Theorem A. Suppose that $\{f_1, f_2, 1\}$ is a linearly independent subset of $V(M^n)$. Since $M^n$ is compact, $M^n$ is conformally diffeomorphic to $S^n([10])$ and there exist exactly two critical points $p_1, q_1$ of $f_1$. Let $p_2$ be a critical point of $f_2$ and let $d = d(p_1, p_2)$ and $l = d(p_1, q_1)$. Then from Lemma 4.1 we have $0 < d < l$. Consider the geodesic $\gamma(t)$ with $\gamma(0) = p_1, \gamma(d) = p_2$, then we have $\gamma(l) = q_1([10])$.

Note that the geodesic sphere $G(p_2, d)$ passes through $p_1$ and meets every geodesic sphere $G(p_1, t), 0 < t \leq d$. For a fixed $t_0 \in (0, d]$ let $q$ be a point in $G(p_2, d) \cap G(p_1, t_0)$ and let $\eta_1, \eta_2$ be the geodesic from $p_1, p_2$ through $q$, respectively. Then we have $\eta_1(t_0) = \eta_2(d) = q$ and $\{\hat{\eta}_1(t_0), \hat{\eta}_2(d)\}$ is linearly independent. Thus Lemma 3.2 implies that $K_1(t_0) = K_1(\hat{\eta}_2(d) \wedge \hat{\eta}_1(t_0)) = K_2(d)$, where $K_1(t)$ and $K_2(t)$ are the sectional curvature functions corresponding to $f_1$ and $f_2$, respectively. Since $G(p_1, d) = G(q_1, l - d)$, for a fixed $t_0 \in [d, l)$, we may as above prove that $K_1(t_0) = K_2(l - d)$. Hence $K_1(t)$ is a constant $k$ on $(0, l)$ hence on $[0, l]$.

Note that $w_1(t)$ satisfies

$$w''_1(t) + kw_1(t) = 0 \quad \text{with} \quad w_1(0) = w_1(l) = 0.$$ 

Thus $k$ must be positive (say, $1/r^2$), so that we have $w_1(t) = a \sin(t/r), a \in R$. The shape operator $S_{p_1}(\gamma(t))$ of the geodesic sphere $G(p_1, t)$ at $\gamma(t)$ satisfies $S_{p_1}(\gamma(t)) = w_1'(t)/w_1(t) = (1/r) \cot(t/r)$, which implies that the sectional curvature $K_M$ of $M^n$ is $k = 1/r^2([9])$. Since $M^n$ is simply connected, $M^n$ is isometric to the Euclidean sphere $S^n(r)$. □

Proof of Theorem B. Let $\{f_1, \cdots, f_n, 1\}$ be a linearly independent subset of $V(M^n)$. Then Lemma 4.2 implies that there exists an open
dense subset $U$ such that $\{\nabla f_1(p), \ldots, \nabla f_n(p)\}$ is linearly independent for all $p \in U$. By Lemma 3.1 we see that the sectional curvature of $M^n$ is a constant $K$ on $U$ hence on $M^n$. Since $M^n$ is noncompact, $K$ is nonpositive. If there is a nonconstant function $f$ in $V(M^n)$ such that $\nabla f(p) = 0$ for some $p \in M$, then $M^n$ must be simply connected, which implies that $M^n$ is isometric to $R^n$ or $H^n(r)$. Thus the proof is completed.

Now suppose that $\nabla f(p) \neq 0$ for all $p \in M$ and for all nonconstant $f \in V(M^n)$. Then by Lemma 4.3, we see that $M^n$ is isometric to the warped product space $R \times_{w_1} F_1$, where $F_1$ is a level hypersurface of $f_1$. Note that $F_1$ is also complete and connected. And Lemma 4.3 shows that $m(F_1) \geq n$ and $F_1$ is also a warped product space $R \times_{w_2} F_2$, where $w_2$ is the length function of gradient of the restriction $\tilde{f}_2$ of $f_2$ on $F_1$ and $F_2$ is a level hypersurface of $\tilde{f}_2$ in $F_1$.

Inductively, we have the following:

$$M^n = R \times_{w_1} F_1$$
$$= R \times_{w_1} (R \times_{w_2} F_2)$$
$$= R \times_{w_1} (R \times_{w_2} (R \times_{w_3} \cdots (R \times_{w_{n-1}} F_{n-1}) \cdots)).$$

Note that each $F_k = R \times_{w_{k+1}} F_{k+1}$ is a complete connected $(n - k)$-dimensional manifold with $m(F_k) \geq n - k + 1$ and $R \times_{w_{n-1}} F_{n-1}$ is a warped product space with $m(R \times_{w_{n-1}} F_{n-1}) \geq 3$. Since we can prove that $V(R \times_{w_{n-1}} S^1) = \{a \int_0^t w_{n-1}(s) ds + b |a, b \in R\}$, $F_{n-1}$ must be the Euclidean line. This implies that $M^n$ is isometric to $R \times_{w_1} R \times \cdots \times_{w_{n-1}} R$, in particular, $M^n$ is simply connected. This completes the proof. □

References

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