ON THE LAWS OF NILPOTENT POINTED-GROUPS

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Abstract. A pointed-group is an ordered pair \((G, c)\) where \(G\) is a
group and \(c\) is a specific element of \(G\). Thus a pointed-group is a
group together with a distinguish element. The aim of this paper is
to generalize the result proved by R. C. Lyndon in [4], that every
nilpotent group variety is finitely based for its laws.

1. Introduction

In [4] R. C. Lyndon proved that every nilpotent group variety has
a finite basis for its laws. (That is, there is a finite set of laws of
which every law is a consequence). Here we shall examine the analogous
statement for nilpotent pointed-groups, where a pointed-group is a pair
\((G, g)\) consisting of a group \(G\) together with a distinguished element \(g\)
of \(G\). (The idea of pointed-groups comes from the category of pointed-
sets), see for example [6].

By a law of a pointed-group \((G, g)\) we shall mean a word \(w\) of the
free group on the countable set \(\{y, x_1, x_2, \ldots, \ldots\}\) such that \(w\) always
becomes equal to the identity element of \(G\) when \(g\) is substituted for \(y\)
and arbitrary elements of \(G\) are substituted for \(x_1, x_2, \ldots\) (For example,
\([y, x_1]\) is law of \((G, g)\) if \(g\) is central in \(G\) or \(G\) is abelian). Included
among the laws of \((G, g)\) are the laws of the group \(G\), or more precisely,
those words in the variable \(x_1, x_2, \ldots\), which are laws of \(G\), thus the idea
of laws of a pointed-group generalizes the idea of laws of a group and the
purpose of this paper is to generalize the result proved by R. C. Lyndon
in [4], that every variety of nilpotent groups is finitely based. Detailed
information concerning varieties of groups may be found in [7].
A pointed-group may be regarded as a group with an extra nullary operation and therefore it is an algebra in the sense of a universal algebra. The more general concepts related to universal algebras and varieties of algebras are described in [2]. It is useful to note the form of some of these concepts take for pointed-groups.

As mentioned above, the factor algebra of \((G, g)\) is \((G/N, gN)\) where \(N\) is a normal sub-group of \(G\) not necessarily containing \(g\). The Cartesian product of the family \(\{(G_\lambda, g_\lambda) | \lambda \in \Lambda\}\) is \((G, g)\) where \(G\) is the Cartesian product of \(\{G_\lambda | \lambda \in \Lambda\}\) and \(g\) is the element of \(G\) with value \(g_\lambda\) at \(\lambda\) for all \(\lambda \in \Lambda\). A generating set for a pointed-group \((G, g)\) is a subset \(S\) of \(G\) such that \(S \cup \{g\}\) generates \(G\). If \((G, g)\) can be generated by a set with \(n\) or fewer elements, then we shall say that \((G, g)\) is an \(n\)-generator pointed-group. By an endomorphism of \((G, g)\) we mean a homomorphism \(\alpha\) from \((G, g)\) to \((G, g)\) such that \(g\alpha = g\). Moreover, if \((G, g)\) is a pointed-group and \(N\) is a normal sub-group of \(G\), then we say that \(N\) is a pointed admissible sub-group of \((G, g)\) if \(N\alpha \leq N\) for every endomorphism \(\alpha\) of \((G, g)\).

A variety of pointed-groups is the class of all pointed-groups in which the elements of some given set of words are all laws. Equivalently, it is a class closed under the operations of taking factor algebras, sub-algebras, Cartesian products and algebras isomorphic to these. We shall write \(\mathcal{V}\) to denote the variety of pointed-groups.

2. Notations and definitions

We call a pointed-group \((G, g)\) nilpotent of class \(c\) if \(G\) is a nilpotent group of class \(c\). Here we shall discuss varieties of pointed-groups in which every pointed-group is nilpotent. Our aim is to generalize the result proved by R. C. Lyndon in [4], that every variety of nilpotent groups is finitely based.

We shall need some notational definitions in the free pointed-group \((X, y)\) generated by \(x_1, x_2, \ldots\).

Put \(X_{(1)} = X\)

and define for \(c \geq 1\) \(X_{(c+1)} = [X_{(c)}, X]\), i.e., the sub-group of \(X\) generated by the elements of the form \([a, b]\) where \(a \in X_{(c)}\) and \(b \in X\). It is
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It is easy to see that each $X(c)$ is a normal admissible sub-group of $(X, y)$. Also, note that $X(c) = [x_1, x_2, \cdots, x_c](X, y)$ for $c \geq 2$, for example, see Theorem 10.2.1 in [3].

Now for each positive integer $c$, we write $N_c$ to denote the variety of nilpotent pointed-groups defined by the word $[x_1, x_2, \cdots, x_{(c+1)}]$ whose closure is $X_{(c+1)}$. Thus $N_0$ is the variety of all trivial pointed-groups and $N_1$ is the variety of all abelian pointed-groups (we call a pointed-group $(G, g)$ abelian if $G$ is an abelian group).

More generally, $N_c$ is the variety of all nilpotent pointed-groups which are nilpotent of class at most $c$.

A variety $V$ of pointed-groups is called nilpotent if $V \subseteq N_c$ for some $c$. We say that $V$ has a class $c$ if $V \subseteq N_c$ and $V \not\subseteq N_{c+1}$.

We will prove the following theorem which is a generalization of the result in [4].

**THEOREM 2.1.** The laws of any variety of nilpotent pointed-groups are finitely-based.

Now in order to prove the above Theorem 2.1, we need to establish some results.

Now for each positive integer $n$, $\delta_n$ denotes the endomorphism of $(X, y)$ determined by $x_n \delta_n = 1$ and $x_i \delta_n = x_i$, for $i \neq n$. Thus the effect of $\delta_n$ on a word $w$, is to delete all occurrences of $x_n$ from $w$.

Now for all positive integers $i, m$ and $n$, it is easy to check that $x_i \delta_m \delta_n = x_i \delta_n \delta_m$ and $x_i \delta_n^2 = x_i \delta_n$. Thus by Corollary 1.4.6 in [1], we have the relations, $\delta_n \delta_m = \delta_m \delta_n$ and $\delta_n^2 = \delta_n$.

Moreover, if $w$ is any word we write $w(1 - \delta_n)$ to denote $w(w \delta_n)^{-1}$. Thus we have $w = w(1 - \delta_n) w \delta_n$.

Also, we have

\[
w(1 - \delta_m) \delta_n = w(w \delta_m)^{-1} \delta_n \\
= w \delta_n (w \delta_m \delta_n)^{-1} \\
= w \delta_n (w \delta_n \delta_m)^{-1} \\
= w \delta_n (1 - \delta_m)
\]
and

\[ w(1 - \delta_n)\delta_n = w(w\delta_n)^{-1} \delta_n = (w\delta_n)(w\delta_n\delta_n)^{-1} = (w\delta_n)(w\delta_n^2)^{-1} = (w\delta_n)(w\delta_n)^{-1} = 1. \]

Hence we have also the relations \( w(1 - \delta_n)\delta_n = 1 \) and \( w(1 - \delta_m)\delta_n = w\delta_n(1 - \delta_m) \). Note that \( w(1 - \delta_n) \) and \( w\delta_n \) are consequences of \( w \). Thus we have:

**Theorem 2.2.** Let \( T \) be a finite set of positive integers. Let \( w \) be a word in \((X, y)\) such that \( w\delta_t = 1 \) for all \( t \in T \). then \( w \in X_{(n)} \) where \( |T| = n \).

*Proof.* See Corollary 33.38 of [7].  \( \square \)

**Theorem 2.3.** Let \( w \) be an element of \( X_{(n)} \) than there are words \( w_T \) for each subset \( T \) of \( \{1, 2, 3, \cdots, n\} \) such that

(i) \( w \) is a product of the words \( w_T \) in some order.

(ii) Each \( w_T \) is a consequence of \( w \).

(iii) Each \( w_T\delta_t = 1 \) for all \( t \in T \) and also \( w_T \) is a word only in the variables \( y \) and \( x_t \in X \) where \( t \in T \).

*Proof.* Assume that \( w \) is a word in the variables \( y \) and \( x_1, x_2, \cdots, x_n \).

Then consider,

\[ w = w(1 - \delta_1)w\delta_1 \]

But,

\[ w(1 - \delta_1) = w(1 - \delta_1)(1 - \delta_2)w(1 - \delta_1)\delta_2 \]

and

\[ w\delta_1 = w\delta_1(1 - \delta_2)w\delta_1\delta_2 \]
Thus (1) gives
\[ w = w(1 - \delta_1)(1 - \delta_2)w(1 - \delta_1)\delta_2w(1 - \delta_1)w_1\delta_2. \]
i.e.,
\[ w = w(1 - \delta_1)(1 - \delta_2)w(1 - \delta_1)\delta_2w(1 - \delta_2)\delta_1w_1\delta_2. \]
Continuing in this way we get,
\[ w = \prod_{T \in \{1,2,3,\ldots,n\}} w_T \]
where the product is taken over the subsets \( T \) of \( N = \{1, 2, \ldots, n\} \) and where
\[ w_T = w \prod_{u \in T} (1 - \delta_u) \prod_{t' \in N \setminus T} \delta_{t'}. \]
Now apply \( \delta_t \) to \( w_t, t \in T \) we have
\[ w_T\delta_t = \left( w \prod_{u \in T} (1 - \delta_u) \prod_{t' \in N \setminus T} \delta_{t'} \right) \delta_t. \]
But, \( \delta_t \) and \( \delta_{t'} \) commute, also \( \delta_t \) and \( (1 - \delta_u) \) commute and \( w \cdots (1 - \delta_t)\delta_t \cdots = 1 \). Hence it follows that \( w_T\delta_t = 1 \). Since this holds for all \( t \in T \) and also \( w_T \) is a word only in the variables \( y \) and \( x_t \) where \( t \in T \), \( w_T \) has the form
\[ v \prod_{t' \in N \setminus T} \delta_{t'} \]
where,
\[ v = w \prod_{u \in T} (1 - \delta_u). \]
and hence does not contain \( x_{t'}, t' \notin T \). Thus the conditions (i) and (iii) are satisfied. Now condition (ii) is an easy consequence of the properties of \( \delta_n \) and \( (1 - \delta_n) \) described above. Thus the proof is complete. \( \square \)
3. Proof of Theorem 2.1

Suppose $V$ is a variety of nilpotent pointed-groups of class at most $c$, i.e., $V \subseteq \mathcal{N}_c$. Let $V$ be the normal admissible subgroup of $(X, y)$ which defines $V$. Let $w \in V$ and suppose that $w = w(y, x_1, x_2, \ldots, x_n)$. Then by Theorem 2.3, $w$ is equivalent to a set of words $\{w_T\}$ where each $T$ is a finite set of positive integers, with the properties given in the statement of Theorem 2.3.

Now suppose $|T| \geq c + 1$, then by Theorem 2.2, $w_T \in X_{(c+1)}$. Therefore, $w_T$ is a consequence of the word $[x_1, x_2, \ldots, x_{c+1}]$. But if $|T| < c + 1$. Then (by the change of variables) $w_T$ is equivalent to an element of $V \cap X_c = V_c$. Thus in either case, $w_T$ is a consequence of $V_{(c+1)}$. Hence, it follows that $w$ is a consequence of $V_{(c+1)}$. Thus $V$ is equivalent to $V_{(c+1)}$. Now by Lemma 2.4.3 in [1] we have

$$V(X_{(c+1)}, y) = V(x, y) \cap X_{(c+1)},$$

i.e.,

$$V(X_{(c+1)}, y) = V_{(c+1)}.$$

Now since $V \subseteq \mathcal{N}_c$, so we have

$$\mathcal{N}_c(X_{(c+1)}, y) \subseteq V(X_{(c+1)}, y).$$

Thus by Corollary 2.2.9 in [1], $\mathcal{N}_c(X_{(c+1)}, y) = K_{(say)}$, is normal in $X_{(c+1)}$. Thus $K \subseteq V_{(c+1)}$ and therefore, $V_{(c+1)}/K$ is a sub-group of $X_{(c+1)}/K$. But $(X_{(c+1)}/K, yK) \in \mathcal{N}_c$ (by Corollary 2.2.10 in [1]). Therefore, $X_{(c+1)}/K$ is a nilpotent group of class at most $c$. But $X_{(c+1)}$ is a finitely generated nilpotent group. Therefore, $X_{(c+1)}/K$ is a finitely generated nilpotent group.

Now Theorem 9.16 of [5], states that in a finitely generated nilpotent group, every sub-group is finitely generated. Hence it follows that $V_{(c+1)}/K$, being a sub-group of $X_{(c+1)}/K$ is finitely generated. Suppose, $v_1 K, v_2 K, \ldots, v_m K$ are generators of $V_{(c+1)}/K$, i.e.,

$$V_{(c+1)}/K = \langle v_1, K, \ldots, v_m K \rangle$$
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Then,

\[ V_{(c+1)} = \langle v_1, v_2, \cdots, v_m, K \rangle \]

For, if \( V(0) = \langle v_1, \cdots, v_m, K \rangle \) then clearly \( V(0) \) is contained in \( V_{(c+1)} \).
Now let \( v \in V_{(c+1)} \), then \( vK \in V_{(c+1)}/K \). Thus \( vK \) can be written in the form

\[
vK = (v_{i_1}K)^{\epsilon_1}(v_{i_2}K)^{\epsilon_2}\cdots(v_{i_n}K)^{\epsilon_n}
= (v_{i_1}^{\epsilon_1}v_{i_2}^{\epsilon_2}\cdots v_{i_n}^{\epsilon_n})K.
\]

Therefore, \( v \) has the form

\[
v_{i_1}^{\epsilon_1}v_{i_2}^{\epsilon_2}\cdots v_{i_n}^{\epsilon_n}k,
\]

where \( k \in K \).

Hence \( v \in V(0) \), i.e., \( V_{(c+1)} \) is contained in \( V(0) \). Thus \( V_{(c+1)} = V(0) \) as desired. Thus \( V \) is equivalent to \( K \cup \{v_1, v_2, \cdots, v_m\} \). But, \( K = N_c(X_{(c+1)}, y) \leq N_c(X, y) \) and \( N_n(X, y) \) is the set of all consequences of \([x_1, x_2, \cdots, x_{c+1}]\). Therefore, \( V \) is equivalent to the set \( \{v_1, v_2, \cdots, v_m, [x_1, x_2, \cdots, x_{c+1}]\} \). Thus \( V \) is finitely based and the proof of Theorem 2.1 is complete.

References


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