WEAKLY LAGRANGIAN EMBEDDING
AND PRODUCT MANIFOLDS

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ABSTRACT. We investigate when the product of two smooth manifolds admits a weakly Lagrangian embedding. We prove that, if $M^m$ and $N^n$ are smooth manifolds such that $M$ admits a weakly Lagrangian embedding into $C^m$ whose normal bundle has a nowhere vanishing section and $N$ admits a weakly Lagrangian immersion into $C^n$, then $M \times N$ admits a weakly Lagrangian embedding into $C^{m+n}$. As a corollary, we obtain that $S^m \times S^n$ admits a weakly Lagrangian embedding into $C^{m+n}$ if $n = 1, 3$. We investigate the problem of whether $S^m \times S^n$ in general admits a weakly Lagrangian embedding into $C^{m+n}$.

1. Introduction

The notion of weakly Lagrangian embedding was introduced by T. Kawashima ([5]) as a weaker version of Lagrangian embedding. He showed that $S^n$ admits a weakly Lagrangian embedding into $C^n$ if and only if $n = 1, 3$, from which it follows that $S^n$ does not admit any Lagrangian embedding into $C^n$ if $n \neq 1, 3$. In fact, later it has been shown that, for any manifold $M^n$ which admits a Lagrangian embedding into $C^n$, we have $\pi_1(M) \neq 1$ ([2]). Therefore it follows that $S^n$ admits a Lagrangian embedding into $C^n$ only when $n = 1$.

This note investigates when the product of two smooth manifolds admits a weakly Lagrangian embedding. In particular, we have

**Theorem 1.** Let $M$, $N$ be smooth manifolds of dimension $m$, $n$, respectively. Assume that $M$ admits a weakly Lagrangian embedding into $C^m$ whose normal bundle has a nowhere vanishing section and $N$
admits a weakly Lagrangian immersion into $\mathbb{C}^n$. Then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.

In fact, the assumption on the existence of a nowhere vanishing section on the normal bundle is redundant if $M$ is an oriented closed manifold: Let $f : M \to \mathbb{C}^m$ be a weakly Lagrangian embedding. We have that $\nu_f \cong (-1)^{n(n-1)/2}TM$ (Proposition 2.1) and $\chi(M) = 0$ (Lemma 4.1). Thus the Euler characteristic of $\nu_f$ vanishes, which means $\nu_f$ admits a nowhere vanishing section.

As a corollary of Theorem 1, we conclude:

**Theorem 2.** $S^m \times S^n$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$ if $n$ is 1 or 3.

In particular, the above provides more examples, in addition to $S^3$, of manifolds which admits a weakly Lagrangian embedding but not any Lagrangian embedding (see Corollary 3.2 below).

Also we have that $S^m \times S^n$ does not admit any weakly Lagrangian embedding into $\mathbb{C}^{m+n}$ if both $m$ and $n$ are even (see the below of Lemma 4.1). However we don’t know what happens when one of $m, n$ is odd while none of the two is 1 or 3, which is a subject of our ongoing investigation. We will provide a reason why this problem is more difficult in this case in the last section.

2. Basic notions and facts

Two subbundles $\eta_0$ and $\eta_1$ of a vector bundle $\xi$ over a smooth manifold $M$ is said to be homotopic if there exists a subbundle $\tilde{\eta}$ of $\xi \times I$ such that $\tilde{\eta}|_{M \times \{0\}} = \eta_0$ and $\tilde{\eta}|_{M \times \{1\}} = \eta_1$.

A symplectic form on a vector bundle is a nondegenerate two form on it. A vector bundle of finite rank is referred to as a Lagrangian vector bundle if it is considered with a fixed symplectic two form. Note that a Lagrangian vector bundle should be of even rank. A subbundle $\eta$ of a Lagrangian vector bundle $\xi$ is a Lagrangian subbundle if $2 (\text{rank } \eta) = \text{rank } \xi$ and the restriction of the symplectic form to $\eta$ is the zero form. A subbundle $\eta$ of a symplectic vector bundle $\xi$ is called a weakly Lagrangian subbundle if $\eta$ is homotopic to a Lagrangian subbundle of $\xi$.  

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Now let \( f : L \to M \) be an embedding (resp. immersion) of a smooth manifold \( L \) into a symplectic manifold \( M \) with a symplectic structure \( \omega \). We call \( f \) a Lagrangian embedding (resp. immersion) if the tangent bundle \( TL \) of \( L \) is a Lagrangian subbundle of the symplectic vector bundle \( f^*TM \) (with the symplectic form \( f^*\omega \)). Similarly, \( f \) is a weakly Lagrangian embedding (resp. immersion) if \( TL \) is a weakly Lagrangian subbundle of \( f^*TM \).

We will consider \( \mathbb{C}^n \) with the usual symplectic structure. A Lagrangian embedding or a weakly Lagrangian embedding will be understood as 'into \( \mathbb{C}^n \)' unless otherwise specified.

Note that the notion of weakly Lagrangian embedding (resp. immersion) is invariant under regular homotopy. That is, if \( f_0 \) and \( f_1 \) are embeddings (resp. immersions) homotopic through embeddings (resp. immersions) and \( f_0 \) is a weakly Lagrangian embedding (resp. immersion), then \( f_1 \) is also such.

We recall some basic properties of a weakly Lagrangian embedding.

**Proposition 2.1.** For a weakly Lagrangian embedding \( f : L^n \to M^{2n} \), from an oriented manifold \( L \), the followings hold

i) \( \nu(f) \cong (-1)^{n(n-1)/2} TL \), as oriented vector bundles, where \( \nu(f) \) is the normal bundle of \( f \) with orientation defined in the usual way.

ii) If \( L \) is a closed manifold and \( a = f_*([L]) \in H_n(M, Z) \), then we have

\[
a \cdot a = (-1)^{n(n-1)} \chi(L)
\]

where \([L] \in H_n(L, Z)\) denotes the fundamental class and \( a \cdot a \) is the Kronecker index \( \langle Da, a \rangle \) with \( D \) denoting the Poincaré isomorphism \( H_n(M, Z) \to H^n_{comp}(M, Z) \).

The proof is a copy of that of Proposition 2, [5]. We note that i) above is true even if \( f \) is only a weakly Lagrangian immersion. On the other hand, we need the condition that \( f \) is an embedding for ii) above since in this case we make use of the normal neighborhood of \( f(L) \subset M \), which is impossible if \( f \) is just an immersion.

3. Proofs of Theorem 1, 2

The following is the key lemma to prove Theorem 1.
Lemma 3.1. Assume $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ are maps between smooth manifolds such that i) $f$ is an embedding whose normal bundle has a nowhere vanishing section and ii) $g$ is an immersion. Then $f \times g : M \times N \to P \times Q$ is regularly homotopic to an embedding.

Proof. We may assume that $g$ is completely regular (cf. [1]). Let $y_1, y_2, \cdots$ and $z_1, z_2, \cdots$ be distinct points in $N$ such that $g(y_i) = g(z_i)$, $i = 1, 2, \cdots$. Note that such points appear discretely.

We may construct (for example, using the exponential map) neighborhoods $U_1, U_2, \cdots$ of $y_1, y_2, \cdots$ which are diffeomorphic to the closed disc $D^n$ and such that $U_i \cap U_j = \emptyset$ if $i \neq j$ and $U_i \cap \{y_1, y_2, \cdots, z_1, z_2, \cdots\} = \{y_i\}$, $i = 1, 2, \cdots$.

Let $\delta : N \to [0, 1]$ be a smooth function such that $\delta(y_i) = 1$, $i = 1, 2, \cdots$ and $\delta(N - \cup_{i=1,2,\cdots} U_i) = \{0\}$.

Note that the existence of nowhere vanishing section of the normal bundle is equivalent to the existence of a smooth embedding $F : M \times [0, 1] \to P$ such that $F(x, 0) = f(x)$.

Now consider the map

$$H : M \times N \times [0, 1] \to P \times Q$$

defined by $H(x, y, t) = (F(x, t\delta(y)), g(y))$.

It is straightforward to see that for each $t \in [0, 1]$, $H_t : M \times N \to P \times Q$ is an immersion. Thus $H_0, H_1$ are regularly homotopic to each other.

We show that $H_1$ is an embedding as follows: Assume that $H_1(x, y) = H_1(x', y')$, that is, $F(x, \delta(y)) = F(x', \delta(y'))$ and $g(y) = g(y')$, while $(x, y) \neq (x', y')$. If $y = y'$, then we have $F(x, \delta(y)) = F(x', \delta(y))$ and we may conclude $x = x'$ since $F$ is an embedding. Therefore, we have $y \neq y'$. Now, by assumption on $g$, $g(y) = g(y')$ implies that $y = y_i, y' = z_i$ (or $y = z_i, y' = y_i$) for some $i$. But then we have $\delta(y_i) = 1$, $\delta(z_i) = 0$ and $F(x, \delta(y)) = F(x', \delta(y'))$ is impossible since $F$ is an embedding. This proves the Lemma. \(\square\)

As corollaries of the previous lemma, we obtain

Proof of Theorem 1. Let $f : M \to \mathbb{C}^m, g : N \to \mathbb{C}^n$ be the weakly Lagrangian embedding and the weakly Lagrangian immersion, respectively. Then $f \times g : M \times N \to \mathbb{C}^m \times \mathbb{C}^n = \mathbb{C}^{m+n}$ is regularly homotopic
to an embedding by the previous lemma. Since being a weakly Lagrangian immersion is invariant under regular homotopy, the proof is complete. □

Proof of Theorem 2. According to Kawashima ([5]), $S^n$ admits a weakly Lagrangian embedding if and only if $n = 1, 3$. Also according to Weinstein ([6]), $S^n$ admits a Lagrangian immersion for any natural number $n$. □

Corollary 3.2. $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ admits a weakly Lagrangian embedding into $C^{n_1+n_2+\cdots+n_k}$ if $n_i = 1$ or 3 for some $i = 1, 2, \cdots, k$.

Note that $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ admits a weakly Lagrangian embedding into $C^{n_1+n_2+\cdots+n_k}$, but it does not admit any Lagrangian embedding into $C^{n_1+n_2+\cdots+n_k}$ if $n_i = 3$ for some $i$ and $n_i \neq 1$ for any $i = 1, 2, \cdots k$, since in this case $\pi_1(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) = 1$.

4. The case of $S^m \times S^n$

As a corollary of ii), Proposition 2.1 we have the following.

Lemma 4.1. Let $L$ be an orientable compact smooth $n$-manifold which admits a weakly Lagrangian embedding into $C^n$. Then we have $\chi(L) = 0$.

Lemma 4.1 proves that, if both $m, n$ are even, $S^m \times S^n$ does not admit any weakly Lagrangian embedding since $\chi(S^m \times S^n) \neq 0$. In fact, the same result can also be obtained by i) of Proposition 2.1 since the tangent bundle of $S^m \times S^n$ is non-trivial if both $m, n$ are even, while the normal bundle of any embedding of $S^m \times S^n$ is trivial if $m, n > 1$, which follows from the triviality of the normal bundle of the standard embedding of $S^m \times S^n$ into $C^{m+n}$ and also from the following.

Lemma 4.2. For any simply connected closed smooth $m$-manifold, $m \geq 4$, any two of its embeddings into $C^m$ are isotopic to each other through smooth embeddings.

Note that if two embeddings are isotopic to each other then the normal bundles of them are isomorphic. A proof of Lemma 4.2 is provided.
below in this section. Note that \( S^2 \) is the only simply connected 2-manifold and it does not admit any weakly Lagrangian embedding and also that any compact orientable 3-manifold is parallelizable. Therefore, we may summarize and generalize the discussions so far as follows.

**Proposition 4.3.** Let \( M \) be a simply connected closed smooth \( m \)-manifold which admits an embedding into \( \mathbb{C}^m \) whose normal bundle is trivial. If \( M \) admits a weakly Lagrangian embedding into \( \mathbb{C}^m \), then \( TM \) is trivial.

Note that, if the tangent bundle of a manifold is trivial, its Euler characteristic vanishes even if the converse is not true in general. Therefore we have obtained a sharper condition than the vanishing of the Euler characteristic for \( S^m \times S^n \) to admit a weakly Lagrangian embedding into \( \mathbb{C}^{m+n} \); its tangent bundle should be trivial.

However, we are not lucky enough here as the following holds.

**Fact.** The tangent bundle of \( S^m \times S^n \) is trivial if \( m \) or \( n \) is odd.

Therefore, even if neither of \( m, n \) is 1 nor 3, we cannot conclude that \( S^m \times S^n \) does not admit any weakly Lagrangian embedding into \( \mathbb{C}^{m+n} \) if one of \( m, n \) is odd. The problem is left open.

The above fact follows from the observation below.
Let \( M \) be a smooth \( m \)-manifold such that
i) the tangent bundle \( TM \) is stably trivial and
ii) \( TM \cong \xi + \epsilon^1_M \) for some vector bundle \( \xi \) over \( M \) of rank \( m - 1 \).
Here \( \epsilon^1_M \) means the trivial vector bundle of rank 1 and \( \xi + \epsilon^1_M \) means the Whitney sum.

Let \( N \) denote another smooth \( n \)-manifold whose tangent bundle is stably trivial and consider the product manifold \( M \times N \).

**Observation.** \( T(M \times N) \cong \epsilon^{m+n}_{M \times N} \).

**Proof.** It is well-known that

\[
T(M \times N) \cong TM \times TN.
\]

By the assumption,

\[
TM \times TN \cong (\xi + \epsilon^1_M) \times TN.
\]
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Let \( p_1, p_2 \) denote the projections from \( M \times N \) to \( M, N \), respectively. Then we have

\[
(\xi + \epsilon_M^1) \times TN = p_1^*(\xi + \epsilon_M^1) + p_2^*TN \cong p_1^*\epsilon + p_1^*\epsilon_M^1 + p_2^*TN.
\]

Now it is straightforward to see that

\[
p_1^*\epsilon + p_1^*\epsilon_M^1 + p_2^*TN \cong p_1^*\epsilon + \epsilon_{M \times N}^1 + p_2^*TN \cong p_1^*\epsilon + p_2^*(TN + \epsilon_N^1).
\]

By the assumption,

\[
p_1^*\epsilon + p_2^*(TN + \epsilon_N^1) \cong p_1^*\epsilon + p_2^*(\epsilon_{N}^{n+1}).
\]

Finally, we have the isomorphisms

\[
p_1^*\epsilon + p_2^*(\epsilon_{N}^{n+1}) \cong p_1^*\epsilon + \epsilon_M^{n+1} \cong \epsilon_{M \times N}^{m+n}
\]

which complete the proof. \( \square \)

To provide the postponed proof of Lemma 4.2, we will need the following by A. Haefliger [3].

THEOREM [Haefliger]. Assume \( V, X \) are smooth manifolds of respective dimensions \( n, k \) and assume \( V \) is compact. Suppose \( 2k \geq 3(n+1) \). Let \( f: V \to X \) be a continuous map such that \( f \) is an embedding in a neighborhood of \( \partial X \) and \( f(\partial V) \cap f(V - \partial V) = \phi \). Assume \( \pi_i(f) = 0 \) for \( 0 \leq i \leq 2n - k + 1 \). Then \( f \) is homotopic to an embedding relative to a neighborhood of \( \partial V \).

Also we need the following fact for which we refer to a work by A. Hatcher [4]. (This must be well known, perhaps with a slightly different condition on the dimensions, even if the authors had problem with finding a more appropriate reference.) In the following, a concordance \( F \) between \( f, g : M \to Q \) means a proper embedding \( F : M \times I \to Q \times I \) such that \( F(x, 0) = (f(x), 0) \) and \( F(x, 1) = (g(x), 1) \) for any \( x \in M \) and an isotopy means a homotopy through embeddings.
THEOREM [Hatcher]. Let $Q, M$ be smooth manifolds with respective dimensions $q, m$. Assume there is a concordance $F : M \times I \to Q \times I$ between two embeddings $f, g : M \to Q$ and $q - m \geq 3$, $q \geq 6$. Then $f, g$ are isotopic to each other.

Proof. According to A. Hatcher ([4]), in particular, Remark 3, p. 229 together with the second paragraph of §2), under the given condition, $F$ is homotopic to the concordance $f \times I : M \times I \to Q \times I$ relative to $M \times \{0\}$ through concordances. Now restrict the homotopy at $M \times \{1\} \equiv M$ to obtain the isotopy from $g$ to $f$.

Proof of Lemma 4.2. Let $M$ denote the manifold and $f, g : M \to C^m$ be the two embeddings. Then since $C^m$ is contractible there is a homotopy $H : M \times I \to C^m$ from $f$ to $g$. Let $\bar{H} : M \times I \to C^m \times I$ denote the map defined by $\bar{H}(x, t) = (H(x, t), t)$ for any $(x, t) \in M \times I$.

We apply the above theorem by Haefliger to conclude that $\bar{H}$ is homotopic to a concordance $F : M \times I \to C^n \times I$ rel $M \times \{0, 1\}$. Here a concordance means simply an embedding such that $F^{-1}(X \times \{0, 1\}) = M \times \{0, 1\}$. Note that, since $M$ is simply connected and $C^n$ is contractible, we have $\pi_i(\bar{H}) = \pi_i(f) = 0$ for $i = 0, 1, 2$ and $2(m + 1) - (2m + 1) + 1 = 2$. Also note that $2(2m + 1) \geq 3(m + 1 + 1)$ if $m \geq 4$.

However the concordance $F$ implies the existence of an isotopy from $f$ to $g$ according to the above theorem by A. Hatcher since $2m - m \geq 3$ and $2m \geq 6$ for any $m \geq 4$.

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References

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