Bull. Korean Math. Soc. **35** (1998), No. 2, pp. 195–201

# CONSTANT NEGATIVE SCALAR CURVATURE ON OPEN MANIFOLDS

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Abstract. We let $(M, g)$ be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature $S$, which is close to $-1$. We show the existence of a conformal metric $\overline{g}$, near to $g$, whose scalar curvature $\overline{S} = -1$ by gluing solutions of the corresponding partial differential equation on each bounded subsets $K_i$ with $\cup K_i = M$.

## 1. Introduction

Let $(M, g)$ be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature $S$. In this paper, we study sufficient conditions for $(M, g)$ to admit a conformal metric $\overline{g} = u^{4/(n-2)}g$, near to $g$, whose scalar curvature $\overline{S} = -1$. This problem is equivalent to finding a smooth positive solution $u$, which is close to 1, of the following partial differential equation:

$$\text{(A)} \qquad -c_n \Delta u + Su = -u^{(n+2)/(n-2)},$$

where $c_n = 4(n-1)/(n-2)$.

To state our theorem, we introduce a notation:

$$Q(M, g) \equiv \inf_{u \in C_0^\infty(M)} \frac{\int_M |\nabla u|^2 + \frac{n-2}{4(n-1)} S u^2 \, dV_g}{(\int_M u^{2n/(n-2)} \, dV_g)^{(n-2)/n}},$$

Received October 17, 1997. Revised November 19, 1997.
1991 Mathematics Subject Classification: 53C21, 58G30.
Key words and phrases: scalar curvature, complete manifolds, conformal metric.
Author is supported in part by KOSEF 96070102013 and BSRI 97-1419 the Ministry of Education.

which is a conformal invariant. Note that $Q(M,g) \leq Q(S^n, g_0)$, where $g_0$ is the standard metric on $S^n$. Any locally conformally flat open Riemannian manifold satisfies $Q(M,g) = Q(S^n, g_0)$ (see [5]). Now we state our Theorem.

THEOREM 1. *Let $(M,g)$ be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature $S$ and infinite volume. Assume that $Q(M,g)$ is finite and $\int_M |S'| + |S'|^{n/2}\, dV_g < \infty$, where $S' = S+1$. Then, there exists a conformal metric $\overline{g} = u^{4/(n-2)} g$ whose scalar curvature is $-1$. Moreover, $u$ is close to $1$ in the following sense:*

$$\text{(B)} \qquad \int_M |\nabla(u-1)|^2 + (u-1)^2 + |u-1|^{2n/(n-2)}\, dV_g < \infty.$$

*If $Q(M,g) \geq 0$ and $u - 1 \in H_1^2(M,g)$, then $u$ is unique and*

$$\int_M |\nabla(u-1)|^2 + (u-1)^2 + |u-1|^{2n/(n-2)}\, dV_g \to 0 \text{ as}$$
$$\int_M |S'| + |S'|^{n/2}\, dV_g \to 0.$$

Developments of conformal changes of metrics on compact manifolds can be found in Aubin [1]. Conformal changes of a metric to a constant negative scalar curvature on noncompact complete Riemannian manifolds have been studied by Aviles and McOwen [2] and Jin [3] using upper and lower solutions. Recently, Li [4] studied this problem when $(M,g)$ satisfies $S \geq -s_0$ for a constant $s_0$, $\lambda_1 > 0$ and other extra conditions, where

$$\lambda_1 \equiv \inf\left\{ \int (c_n |\nabla\phi|^2 + s(x)\phi^2)\, dV_g : \ \phi \in C_0^1(M), \ \int \phi^2\, dV_g = 1 \right\}.$$

In this paper, we study conformal metrics using integrals of perturbed scalar curvature $S' = S + 1$ without using a lower bound of scalar curvature.

## 2. Proof of main results

First we sketch the proof of the existence of a positive solution of (A), which you can find details in [4]. The following existence of a conformal metric on a smooth bounded domain with nonzero boundary data is known (see [4]).

LEMMA 1. *Let $K_i$ is a smooth bounded domain with boundary $\partial K_i$ Suppose that $\psi$ is positive smooth function on $\partial K_i$. Then there exists a positive solution $u$ of* (A) *with $u = \psi$ on $\partial K_i$. Moreover $u$ is unique when $Q(M, g) \geq 0$.*

Assume that there exists a sequence of smooth bounded domains $\{K_i\}$ with $K_i \subset K_{i+1}$ and $\cup K_i = M$. By Lemma 1, there exists a smooth positive solution $u_i$ of (A) on each $K_i$ and $u_i = 1$ on $\partial K_i$. We extend the domain of $u_i$ by defining $u_i = 1$ on the outside of $K_i$. We use the same notation $u_i$ for this extension. Note that the extension $u_i - 1$ is in $H_1^2(M, g)$. We construct a positive solution of (A) on $M$ by gluing solutions $u_i$ of equation (A) on each $K_i$.

Aviles and McOwen [2] showed the following Lemma.

LEMMA 2. *For each compact set $X \subset \Omega$ there exists a constant $C_0$ such that any nonnegative weak solution $u \in H_1^2(\Omega)$ of* (A) *satisfies*

$$\max_{x \in X} u(x) \leq C_0.$$

Using the elliptic estimates and Lemma 2, we have a convergent subsequence $\{u_{ki}\}$ which converges to $u$ in $C^{2,\alpha}$ on each compact subset. Using the maximum principle, we have a positive solution $u$ for (A).

Next we estimate the behavior of solutions of (A). Let $h_i = u_i - 1$, $h = u - 1$, $\alpha = (n+2)/(n-2)$, $X_1 = \{x \in K_i | -1 < h_i(x) < 1\}$ and $X_2 = \{x \in K_i | 1 \leq h_i(x)\}$. Note that $h(x) > -1$.

CLAIM 1. *$\int_M |h_i|^{\alpha+1} dV_g$ is bounded.*

*Proof.* First we give a bound on $\int_M |S'h_i|\, dV_g$. By Young's inequality, there exists a positive constant $C(c_1)$ for a given positive constant $c_1$ with the following:

$$
\begin{aligned}
\int_{K_i} |S'h_i|\, dV_g &\leq \int_{X_1} |S'h_i|\, dV_g + \int_{X_2} |S'h_i|\, dV_g, \\
&\leq \int_{X_1} |S'|\, dV_g + \int_{X_2} |S'h_i^2|\, dV_g, \\
&\leq \int_{X_1} |S'|\, dV_g + \int_{X_2} C(c_1)|S'|^{n/2} + c_1|h_i|^{\alpha+1}\, dV_g, \\
&\leq \int_{M} |S'|\, dV_g + \int_{M} C(c_1)|S'|^{n/2} + c_1|h_i|^{\alpha+1}\, dV_g.
\end{aligned} \tag{1}
$$

Therefore, we have

$$
\int_M |S'h_i|\, dV_g \leq \int_M |S'|\, dV_g + \int_M C(c_1)|S'|^{n/2} + c_1|h_i|^{\alpha+1}\, dV_g. \tag{2}
$$

From the given condition of Theorem 1,

$$
\begin{aligned}
&Q(M,g)\Big(\int_{K_i} |h_i|^{\alpha+1}\, dV_g\Big)^{2/\alpha+1} \\
&\leq \int_{K_i} (-c_n\Delta h_i + Sh_i)h_i\, dV_g, \\
&\leq \int_{K_i} (-c_n\Delta(u_i - 1) + S(u_i - 1))h_i\, dV_g, \\
&\leq \int_{K_i} (-u_i^\alpha - S)h_i\, dV_g, \\
&= \int_{K_i} (-(1 + h_i)^\alpha + 1 - S')h_i\, dV_g.
\end{aligned} \tag{3}
$$

Using the basic inequality, $0 \leq |h_i|^{\alpha+1} \leq \big((1 + h_i)^\alpha - 1\big)h_i$, (1) and

(3), we have

$$
\begin{aligned}
&Q(M,g)\Big(\int_{K_i} |h_i|^{\alpha+1}\, dV_g\Big)^{2/\alpha+1} + \int_{K_i} |h_i|^{\alpha+1}\, dV_g, \\
&\le Q(K_i,g)\Big(\int_{K_i} |h_i|^{\alpha+1}\, dV_g\Big)^{2/\alpha+1} + \int_{K_i} ((1+h_i)^\alpha - 1)h_i\, dV_g, \\
&\le \int_{K_i} -S' h_i\, dV_g, \\
&\le \int_{K_i} |S'|\, dV_g + \int_{K_i} C(c_1)|S'|^{n/2} + c_1|h_i|^{\alpha+1}\, dV_g.
\end{aligned} \tag{4}
$$

Taking $c_1 < 1$ in the above, we have:

$$
\begin{aligned}
\int_{K_i} |h_i|^{\alpha+1}\, dV_g &\le C\Big(\int_{K_i} |S'|\, dV_g + \int_{K_i} |S'|^{n/2}\, dV_g + C_2\Big), \\
&\le C\Big(\int_{M} |S'|\, dV_g + \int_{M} |S'|^{n/2}\, dV_g + C_2\Big).
\end{aligned} \tag{5}
$$

where $C$ and $C_2$ are positive constants independent of $i$. Note that we take $C_2 = 0$ when $Q(M,g) \ge 0$. Therefore we have

$$
\int_{M} |h_i|^{\alpha+1}\, dV_g \le C\Big(\int_{M} |S'|\, dV_g + \int_{M} |S'|^{n/2}\, dV_g + C_2\Big). \tag{6}
$$

$\square$

CLAIM 2. *$\int_M |\nabla h_i|^2\, dV_g$ is bounded.*

*Proof.*

$$
\begin{aligned}
\int_{K_i} c_n|\nabla h_i|^2\, dV_g &= \int_{K_i} -c_n h_i \Delta h_i\, dV_g, \\
&= \int_{K_i} -c_n h_i \Delta u_i\, dV_g, \\
&\le \int_{K_i} (-S u_i - u_i^\alpha) h_i\, dV_g, \\
&\le \int_{K_i} ((1-S')(1+h_i) - (1+h_i)^\alpha) h_i\, dV_g.
\end{aligned} \tag{7}
$$

From (7),

$$
\begin{aligned}
&\int_{K_i} c_n|\nabla h_i|^2 + \big((1+h_i)^\alpha - (1+h_i)\big)h_i\, dV_g \\
&\le \int_{K_i} -S'(1+h_i)h_i\, dV_g, \\
&\le \int_{X_1} |S'(1+h_i)h_i|\, dV_g + \int_{X_2} |S'(1+h_i)h_i|\, dV_g, \qquad (8)\\
&\le 2\int_{X_1} |S'|\, dV_g + 2\int_{X_2} |S'h_i^2|\, dV_g, \\
&\le 2\int_M |S'|\, dV_g + 2\Big(\int_M |S'|^{n/2}\, dV_g\Big)^{2/n}\Big(\int_M |h_i|^{\alpha+1}\, dV_g\Big)^{2/\alpha+1}.
\end{aligned}
$$

By Claim 1, we conclude that $\int_M |\nabla h_i|^2\, dV_g < \infty$ since $\big((1+h_i)^\alpha - (1+h_i)\big)h_i \ge 0$ for $h_i > -1$. □

CLAIM 3. *$\int_M h_i^2\, dV_g$ is bounded.*

*Proof.*

$$
\begin{aligned}
Q(M,g)\Big(\int_{K_i} |h_i|^{\alpha+1}\, dV_g\Big)^{2/\alpha+1} &\le \int_{K_i} (-c_n\Delta h_i + Sh_i)h_i\, dV_g, \\
&\le \int_{K_i} c_n|\nabla h_i|^2 + (-1+S')h_i^2\, dV_g. \qquad (9)
\end{aligned}
$$

From (9),

$$
\begin{aligned}
&Q(M,g)\Big(\int_{K_i} |h_i|^{\alpha+1}\, dV_g\Big)^{2/\alpha+1} + \int_{K_i} h_i^2\, dV_g, \\
&\le \int_{K_i} c_n|\nabla h_i|^2 + S'h_i^2\, dV_g, \\
&\le \int_M c_n|\nabla h_i|^2 dV_g + \Big(\int_M |S'|^{n/2}\, dV_g\Big)^{2/n}\Big(\int_M |h_i|^{\alpha+1}\, dV_g\Big)^{1/\alpha+1}.
\end{aligned}
$$

Finally, we have

$$\int_M h_i^2\, dV_g \le \int_M c_n |\nabla h_i|^2 dV_g + \Big( |Q(M,g)| \big( \int_M |h_i|^{\alpha+1} dV_g \big)^{1/\alpha+1}$$

$$+ \big( \int_M |S'|^{n/2}\, dV_g \big)^{2/n} \Big) \Big( \int_M |h_i|^{\alpha+1}\, dV_g \Big)^{1/\alpha+1}.$$

By Claim 1 and Claim 2, we have a bound in the right hand side of the above equation. □

From the above Claims and $u_i = 1 + h_i \to u$, we have (B) in Theorem 1. Since volume of $(M,g)$ is infinite and (B), $u$ can not be identically zero. By the maximum principle, we have a positive solution of (A). The second part of Theorem 1 comes from the fact that we can take $C_2 = 0$ in (5) and the maximum principle.

REMARK. We can weaken the volume condition on Theorem 1. There exists a positive constant $C_3$ such that Theorem 1 holds when volume of $M$ is greater than $C_3$. This comes from Claim 1.

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