MAXIMAL INEQUALITIES WITH AN APPLICATION  
TO THE WEAK CONVERGENCE FOR  
2-PARAMETER ARRAYS OF POSITIVELY DEPENDENT RANDOM VARIABLES

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ABSTRACT. We derive maximal inequalities of linearly positive quadrant dependent (LPQD) random variables for $d = 1, 2$. With an application we also obtain the weak convergence for 2-parameter arrays of LPQD random variables.

1. Introduction

Lehmann(1966) introduced a simple and natural definition of positive dependence: Two random variables $X$ and $Y$ are said to be positive quadrant dependent (PQD) if for any real $x, y$ $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ and a sequence $\{X_j : j \geq 1\}$ of random variables is called pairwise PQD if for any real $r_i, r_j$ and $i \neq j$, $P(X_i > r_i, X_j > r_j) \geq P(X_i > r_i)P(X_j > r_j)$. A much stronger concept than pairwise PQD was considered by Esary, Proschan, and Walkup(1967): A finite family of random variables is said to be associated if

$$\text{Cov}(f(X_1, \ldots, X_m), g(X_1, \ldots, X_m)) \geq 0$$

for any coordinatewise nondecreasing functions $f$ and $g$ on $\mathbb{R}^m$ whenever the covariance exists. A family of infinite number of random variables is associated if every finite subfamily is associated. A sequence $\{X_j : j \geq 1\}$

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of random variables is said to be linearly positive quadrant dependent (LPQD) if for any disjoint \( A, B \) and positive \( r_j \)’s

\[
\sum_{i \in A} r_i X_i \text{ and } \sum_{j \in B} r_j X_j \text{ are PQD.}
\]

Let us remark that this concept of positive dependence is between pairwise PQD and association and it is well known (see, for example, [11, p131]) that neither pairwise PQD nor LPQD nor association implies the others. A \( d \)-parameter array \( \{X_{\hat{j}} : \hat{j} \in Z^d\} \) of random variables is called stationary if for all \( m \) and for all \( j, k_1, \cdots, k_m \in Z^d(X_{k_1}, \cdots, X_{k_m}) \) has the same distribution as \( (X_{j+k_1}, \cdots, X_{j+k_m}) \). Throughout this paper we will deal with the stationary arrays. Newman(1980) derived a weak convergence of finite dimensional distributions for the \( d \)-parameter arrays of associated random variables. When \( d = 1 \) and 2 this convergence can be strengthened to yield the invariance principles (see [12] and [13]). Dabrowski(1985) proved a functional law of iterated logarithm for \( d = 1 \). Burton and Kim(1988) showed an invariance principle for the \( d \)-parameter arrays under a stronger moment condition than Newman and Wright (1982). In the nonstationary associated case, Birkel(1988) obtained the invariance principle for \( d = 1 \) under the second moment condition and Kim(1996) derived an invariance principle for \( d \)-parameter arrays under \( 2 + \delta \) moment condition. For linearly positive quadrant dependence Birkel(1993) proved a functional central limit theorem for a nonstationary LPQD sequence under \( 2 + \delta \) moment condition and Kim and Seo(1996) extended it to the \( d \)-parameter arrays.

Newman(1980) had already mentioned that the weak convergence of finite dimensional distributions for stationary \( d \)-parameter arrays of associated random variables, still holds for the LPQD case instead of association (see [10, p122]) under second moment condition. The following theorem is a modified version of Theorem 2 in Newman(1980):

**Theorem A (Newman, 1980.)** Let \( \{X_{\hat{j}} : \hat{j} = (j_1, \cdots, j_d) \in Z^d\} \) be a stationary \( d \)-parameter array of LPQD random variables with \( EX_{\hat{j}} = 0, EX_{\hat{j}}^2 < \infty \). Assume

\[
0 < \sigma^2 = \sum_{\hat{j} \in Z^d} Cov(X_0, X_{\hat{j}}) < \infty.
\]
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For \( t \in [0, 1] \) define

\[
W_n(t) = n^{-\frac{d}{2}} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{j},
\]

where \([\cdot]\) denotes the usual greatest integer function, and let \( W_n(t) \) be the \( d \)-parameter Wiener process, a mean zero Gaussian process with \( \text{Cov}(W(t), W(s)) = \sigma^2 \prod_{i=1}^{d} \min(t_i, s_i) \). Then the finite dimensional distributions of \( W_n \) converges in distribution to those of \( W \).

By requiring the stronger concept of association instead of LPQD Newman and Wright (1982) obtained a weak convergence for a stationary 2-parameter array of associated random variables.

**Theorem B** (Newman, Wright 1982.) Let \( \{X_{j} : j \in \mathbb{Z}^2\} \) be a stationary 2-parameter array of associated random variables with \( E X_{j} = 0, \ E X_{j}^2 < \infty \). Let \( W_n(t_1, t_2) = n^{-1} \sum_{j_1=1}^{[nt_1]} \sum_{j_2=1}^{[nt_2]} X_{j} \) and let \( W(t) \) be the 2-parameter Wiener process, a mean zero Gaussian process with \( \text{Cov}(W(t), W(s)) = \sigma^2 \prod_{i=1}^{2} \min(t_i, s_i) \). Assume that (3) holds for \( d = 2 \). Then \( W_n(t) \) converges weakly to \( W(t) \).

In this paper, we show maximal inequalities of LPQD random variables for \( d = 1 \) and 2 and obtain a weak convergence for 2-parameter arrays of LPQD random variables by using the maximal inequality together with Theorem A.

In Section 2 maximal inequalities for one dimensional LPQD random variables are presented and in Section 3 maximal inequalities for the 2-dimensional arrays are derived by applying the methods of the proofs of the maximal inequalities for associated random variables in Newman and Wright (1982). We also obtain the weak convergence for a stationary 2-parameter array of LPQD random variables by using the maximal inequality and Theorem A in Section 4.

The problem of proving the maximal inequality of associated random variables for \( d \geq 3 \) is presently an open question(see [13]). Hence to show the maximal inequality of LPQD random variables for \( d \geq 3 \) is also open.
2. Preliminaries

We start this section with introducing Lemma 1 of Lehmann (1966).

**Lemma 2.1.** (Lehmann, 1966.) If $X_1$ and $X_2$ are PQD then $\text{Cov}(f(X_1) \ g(X_2)) \geq 0$ for any nondecreasing (nonincreasing) functions $f, g$.

Let $S_n^* = \max(S_1, \ldots, S_n)$ and $S_n^{**} = \min(S_1, \ldots, S_n)$, where $S_n = X_1 + \cdots + X_n$.

**Theorem 2.1.** Let $\{X_j : j \geq 1\}$ be a sequence of stationary LPQD random variables with $EX_j = 0$ and let $m$ be a nonnegative and nondecreasing function with $m(0) = 0$. Then for any $n$

\begin{align*}
(5) & \quad E \left( \int_0^{S_n^*} u dm(u) \right) \leq E(S_n m(S_n^*)), \\
(6) & \quad E \left( \int_0^{S_n^{**}} u dm(u) \right) \leq E(S_n m(S_n^{**})),
\end{align*}

and for any $\lambda > 0$,

\begin{align*}
(7) & \quad \lambda P(S_n^* \geq \lambda) \leq \int_{\{S_n^* \geq \lambda\}} S_n dP, \\
(8) & \quad \lambda P(S_n^{**} \geq \lambda) \leq \int_{\{S_n^{**} \geq \lambda\}} S_n dP.
\end{align*}

**Proof.** Following the lines of the proof of Theorem 3 of Newman and Wright (1982) we will prove this theorem. Let $S_0^* = 0$. Then

\begin{align*}
(9) & \quad S_n m(S_n^*) = \sum_{k=0}^{n-1} S_{k+1}(m(S_{k+1}^*) - m(S_k^*)) + \sum_{k=1}^{n-1} (S_{k+1} - S_k) m(S_k^*).
\end{align*}

Note that either $S_k^* = S_{k+1}^*$ or $S_{k+1} = S_{k+1}^* > S_k^*$. Thus for any $k$,

\begin{align*}
(10) & \quad S_{k+1}(m(S_{k+1}^*) - m(S_k^*)) = S_{k+1}(m(S_{k+1}^*) - m(S_k^*)) \\
& \quad \geq \int_{S_k^*}^{S_{k+1}^*} u \ dm(u).
\end{align*}
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From (9) and (10) we obtain

\[
S_n m(S_n^*) \geq \sum_{k=0}^{n-1} \int_{S_n^*}^{S_{k+1}^*} u \, dm(u) + \sum_{k=1}^{n-1} ((S_{k+1} - S_k)m(S_k^*))
\]

\[
= \int_0^{S_n^*} u \, dm(u) + \sum_{k=1}^{n-1} ((S_{k+1} - S_k)m(S_k^*)).
\]

Next we note that \(X_{k+1}\) and \(S_j, 1 \leq j \leq k\) are \(PQD\) by the definition of \(LPQD\) (see (2)) and that \(m(S_k^*)\) is a nondecreasing function of \(S_j, 1 \leq j \leq k\), since \(S_k^*\) is a nondecreasing function of \(S_j, 1 \leq j \leq k\). Thus by Lemma 2.1 and assumption \(EX_j = 0\)

\[
E((S_{k+1} - S_k)m(S_k^*)) = Cov(X_{k+1}, m(S_k^*)) \geq 0.
\]

By taking the expection of (11) and using (12) we obtain

\[
E(S_n m(S_n^*)) \geq E(\int_0^{S_n^*} u \, dm(u)),
\]

which yields (5). Take \(m(u) = 1_{\{u \geq \lambda\}}\) in (5). Then it follows from (5) that

\[
L.H.S \text{ of (5) } = E(\int_0^{S_n^*} u \, dm(u)) = E[u1_{\{u \geq \lambda\}}]_{S_n^*} = E[S_n^*1_{\{S_n^* \geq \lambda\}}]
\]

\[
\geq E[\lambda 1_{\{S_n^* \geq \lambda\}}] = \lambda P\{S_n^* \geq \lambda\},
\]

and

\[
R.H.S \text{ of (5) } = E(S_n m(S_n^*)) = E[S_n 1_{\{S_n \geq \lambda\}}]
\]

\[
= \int_{\{S_n \geq \lambda\}} S_n \, dP.
\]

Similarly, (6) and (8) are proved. \(\square\)

The following corollary is corresponding to Corollary 4 of Newman and Wright (1982).

Corollary 2.3. Let \(\{X_j : j \geq 1\}\) be a stationary sequence of \(LPQD\) random variables with \(EX_j = 0\). Then

\[
E((S_n^* - S_n)^2) \leq E(S_n^2)
\]

\[
E((S_n^{**} - S_n)^2) \leq E(S_n^2)
\]
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Proof. By taking $m(u) = u1_{\{u \geq 0\}}$ in (5) we have $E(S_n^*)/2 \leq E(S_n S_n^*)$, which yields $ES_n^* - 2(ES_n S_n^*) + ES_n^2 \leq ES_n^2$ and thus (15) holds. Similarly, (16) is obtained.

Applying the methods of the proofs in Corollary 5 and Corollary 6 of [10] we will prove Corollary 2.4 and Corollary 2.5.

COROLLARY 2.4. Let $\{X_j : j \geq 1\}$ be a stationary sequence of LPQD random variables with $EX_j = 0$. Then

\begin{align}
E(S_n^{*2}) & \leq E(S_n^2), \\
E(S_n^{*2}) & \leq E(S_n^2).
\end{align}

Proof. Define $T_1 = 0$ and $T_k = \sum_{i=n-k+2}^n X_i$ for $k = 2, 3, \ldots, (n+1)$ and let $T_n^{**} = \min(T_1, \ldots, T_n)$. Clearly, $T_2 = X_n, T_3 = X_n + X_{n-1}, \ldots, T_n = X_n + \cdots + X_2, T_{n+1} = X_n + \cdots + X_1$. Since $\{X_n, X_{n-1}, \ldots, X_2, X_1\}$ is a sequence of LPQD random variables by (6) with $m(u) = u1_{\{u \geq 0\}}$

\begin{align}
E(T_n^{**2}/2) & \leq E(T_nT_n^{**}).
\end{align}

Note that $X_1$ and $T_j, 1 \geq j \geq n$ are PQD by definition of LPQD(see (2)) and that $T_n^{**}$ is a nondecreasing function of $T_j, 1 \leq j \leq n$. Hence $E(X_1 T_n^{**}) = Cov(X_1, T_n^{**}) \geq 0$ by Lemma 2.1 and

\begin{align}
E(T_nT_n^{**}) & \leq E(T_nT_n^{**}) + E(X_1 T_n^{**}) \\
& = E(T_{n+1} T_n^{**}).
\end{align}

From (19) and (20) we obtain $E(T_{n+1} - T_n^{**})^2 \leq E(T_{n+1}^2)$, which is the same as (17) since $T_{n+1} = S_n$ and

\begin{align}
T_{n+1} - T_n^{**} & = \max(T_{n+1} - T_n, T_{n+1} - T_{n-1}, \ldots, T_{n+1} - T_1) \\
& = \max(S_1, S_2, \ldots, S_n) = S_n^*.
\end{align}

Next, to show (18) let $T_n^* = \max(T_1, \cdots, T_n)$ and take $m(u) = u1_{\{u \geq 0\}}$ in (6). Then

\begin{align}
E(T_n^{*2}/2) & \leq E(T_n T_n^*).
\end{align}

Note that $X_1$ and $T_j, 1 \leq j \leq n$ are PQD by definition of LPQD and that $T_n^*$ is a nondecreasing function of $T_j$. Hence $E(X_1 T_n^*) = Cov(X_1, T_n^*) \geq 0$
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by Lemma 2.1 and

\[(22) \quad E(T_nT_n^*) \leq E(T_nT_n^*) + E(X_1T_n^*) = E(T_{n+1}T_n^*). \]

From (21) and (22) we obtain \(E(T_{n+1} - T_n^*)^2 \leq E(T_{n+1}^2)\), which is the same as (18) since \(T_{n+1} = S_n\) and \(T_{n+1} - T_n^* = \min(S_1, S_2, \ldots, S_n) = S_n^{**}. \)

The next corollary will be used to obtain the maximal inequality for \(d = 2\) in Section 3 below.

**Corollary 2.5.** Let \(\{X_j : j \geq 1\}\) be a stationary sequence of LPQD random variables with \(EX_j = 0\). Then for \(0 \leq \lambda_1 < \lambda_2\),

\[(23) \quad P(S_n^* \geq \lambda_2) \leq \left(\frac{ES_n^2}{(\lambda_2 - \lambda_1)^2}\right)^{\frac{1}{2}} (P(S_n \geq \lambda_1))^{\frac{1}{2}}, \]

\[(24) \quad P(\max(|S_1|, \ldots, |S_n|) \geq \lambda_2) \leq \sqrt{2} \left(\frac{ES_n^2}{(\lambda_2 - \lambda_1)^2}\right)^{\frac{1}{2}} (P(|S_n| \geq \lambda_1))^{\frac{1}{2}}. \]

**Proof.** Starting from (7) with \(\lambda = \lambda_2\) we have

\[\lambda_2 P(S_n^* \geq \lambda_2) \leq \int_{\{S_n^* \geq \lambda_2\}} S_n dP \leq \int_{\{S_n \geq \lambda_1\}} S_n dP + \int_{\{S_n^* \geq \lambda_2, S_n < \lambda_1\}} S_n dP \]

\[\leq \int_{\{S_n \geq \lambda_1\}} S_n dP + \lambda_1 P(S_n^* \geq \lambda_2), \]

which immediately yields

\[(25) \quad P(S_n^* \geq \lambda_2) \leq (\lambda_2 - \lambda_1)^{-1} E(S_n 1_{\{S_n \geq \lambda_1\}}). \]

The Cauchy-Schwarz inequality applied to the right hand side of (25) then yields (23). To obtain (24) we add to (23) the analogous inequality with all \(X_i\)'s replaced by their negatives (which also are LPQD) and use the fact that for \(A, B \geq 0, \sqrt{A} + \sqrt{B} \leq \sqrt{2(A + B)}\). \(\square\)
3. A maximal inequality of 2-parameter arrays

Throughout this section we deal with a stationary 2-parameter array \( \{X_{i,j} : j = (j_1, j_2) \in \mathbb{Z}^2 \} \) of \( LPQD \) random variables with \( EX_{i,j} = 0, EX_{i,j}^2 < \infty \), and the partial sum

\[
S_{j} = S_{(j_1,j_2)} = \sum_{i=1}^{j_1} \sum_{k=1}^{j_2} X_{(i,k)}.
\]

We also define for \( m, n \geq 1 \),

\[
S^*_{(m,n)} = \max \{ S_{j} : 1 \leq j_1 \leq m, 1 \leq j_2 \leq n \}.
\]

In order to strengthen Theorem A to obtain weak convergence for \( d = 2 \), we need a maximal inequality which controls the tail of \( S^*_{(m,n)} \) in terms of the tail of \( S_{(m,n)} \) as done for \( d = 1 \) by (23). Our \( d = 2 \) result will in fact be based on (23) and the key step is the following lemma; our approach is modelled after previous work on maximal inequalities for 2-parameter associated random variables.

**Lemma 3.1.** For fixed \( m \), let

\[
\overline{S}_j = \max \{ S_{(k,j)} : 1 \leq k \leq m \},
\]

(26)

\[
\overline{S}^*_j = \max (\overline{S}_1, \ldots, \overline{S}_j).
\]

(27)

Then for all nonnegative and nondecreasing function \( m \) with \( m(0) \),

(28)

\[
E(\overline{S}_{j+1} - \overline{S}_j)m(\overline{S}_{n}^*) \geq 0.
\]

**Proof.** Note that \( \overline{S}^*_j = S^*_{(m,j)} \). We apply the arguments of the proof of (37) of Lemma 9 in Newman and Wright (1982). Define, \( K_j \) by

\[
K_j = \min \{ k : S_{(k,j)} = \max (S_{(1,j)}, \ldots, S_{(m,j)}) \}.
\]

Then

\[
\overline{S}_{j+1} - \overline{S}_j = \overline{S}_{j+1} - \overline{S}_{(K_j,j)} \geq S_{(K_j,j+1)} - S_{(K_j,j)} = \sum_{k=1}^{K_j} X_{(k,j+1)},
\]

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and so since \( m(\overline{S}_j^*) \geq 0 \) we have

\[
(29) \quad (\overline{S}_{j+1} - \overline{S}_j) m(\overline{S}_j^*) \geq \sum_{k=1}^{K_j} X_{(k,j+1)} m(\overline{S}_j^*).
\]

Taking expectation on both side of (29)

\[
(30) \quad E((\overline{S}_{j+1} - \overline{S}_j) m(\overline{S}_j^*)) \geq E \left( \sum_{k=1}^{K_j} X_{(k,j+1)} m(\overline{S}_j^*) \right) = \sum_{k=1}^{K_j} E[X_{(k,j+1)} m(\overline{S}_j^*)] = \sum_{k=1}^{K_j} Cov(X_{(k,j+1)}, m(\overline{S}_j^*)) \geq 0
\]

where we have used the fact that \( EX_{(k,j)} = 0 \) for every \( k \) and \( j \). The non-negativity of \( Cov(X_{(k,j+1)}, m(\overline{S}_j^*)) \) follows from the definition of \( LPQD \) and Lemma 2.1 since \( m(\overline{S}_j^*) \) is a nondecreasing function of \( S_{ij} \), \( 1 \leq i \leq m \) not including \( X_{(k,j+1)} \) and \( X_{(k,j+1)} \) and \( S_j^* \) are \( PQD \) as sums of disjoint subsets of \( X_{ij} \)'s.

**Remark 1.** Lemma 3.1 can not be extended to \( d \geq 3 \) and thus the maximal inequality for \( d \geq 3 \) is not obtained in this way.

From Theorem 2.2 and Lemma 3.1 we obtain the following lemma:

**Lemma 3.2.** Let \( \overline{S}_j \) and \( \overline{S}_j^* \) be as in (26) and (27), respectively. For any nonnegative and nondecreasing function \( m \) with \( m(0) = 0 \) and any \( j \geq 1 \)

\[
(31) \quad E \left( \int_0^{\overline{S}_n} u \, dm(u) \right) \leq E(\overline{S}_j m(\overline{S}_j^*))
\]

and for any \( \lambda > 0 \),

\[
(32) \quad \lambda P(\overline{S}_j^* \geq \lambda) \leq \int_{\{\overline{S}_j \geq \lambda\}} \overline{S}_j dP.
\]
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Proof. We will use the method of the proof of Theorem 2.2 and relation (28). Note that

\begin{equation}
\overline{S}_j m(\overline{S}_j^*) = \sum_{i=0}^{j-1} \overline{S}_{i+1}(m(\overline{S}_{i+1}^*) - m(\overline{S}_i^*)) + \sum_{i=1}^{j-1} (\overline{S}_{i+1} - \overline{S}_i)m(\overline{S}_i^*)
\end{equation}

and that either $\overline{S}_{i+1}^* = \overline{S}_i^*$ or $\overline{S}_{i+1}^* = \overline{S}_{i+1}^*$. Thus

\begin{equation}
\overline{S}_{i+1}(m(\overline{S}_{i+1}^*) - m(\overline{S}_i^*)) = \overline{S}_{i+1}(m(\overline{S}_{i+1}^*) - m(\overline{S}_i^*)) \\
\geq \int_{\overline{S}_i^*} u \, dm(u).
\end{equation}

Let $\overline{S}_o^* = 0$. Then (33) and (34) yield

\begin{equation}
\overline{S}_j m(\overline{S}_j^*) \geq \sum_{i=0}^{j-1} \int_{\overline{S}_i^*} u \, dm(u) + \sum_{i=1}^{j-1} ((\overline{S}_{i+1} - \overline{S}_i)m(\overline{S}_i^*)) \\
= \int_{0}^{\overline{S}_o^*} u \, dm(u) + \sum_{i=1}^{j-1} ((\overline{S}_{i+1} - \overline{S}_i)m(\overline{S}_i^*)�
\end{equation}

By taking the expectation of (35) and using (28) we obtain (31). In (31) by taking $m(u)$ to be the indicator function $1_{\{u \geq \lambda\}}$ (32) follows. \qed

From Corollary 2.5 and (32) we obtain the following lemma.

**Lemma 3.3.** For $0 \leq \lambda_1 < \lambda_2$,

\begin{equation}
P(S_n^* \geq \lambda_2) \leq \left( \frac{E(S_n^2)}{(\lambda_2 - \lambda_1)^2} \right)^{\frac{1}{2}} (P(S_n^* \geq \lambda_1))^{\frac{1}{2}}.
\end{equation}

We will use Corollary 2.4 and Corollary 2.5 and Lemma 3.3 to prove the following theorem, which generalizes (23) and (24) to $d = 2$.

**Theorem 3.4.** For $0 \leq \lambda_1 < \lambda_2$,

\begin{equation}
P(S_{(m,n)}^* \geq \lambda_2) \leq (3^{\frac{3}{2}})(2^{-1}) \left( \frac{ES_{(m,n)}^2}{(\lambda_2 - \lambda_1)^2} \right)^{\frac{3}{4}} (P(S_{(m,n)}^* \geq \lambda_1))^{\frac{1}{4}},
\end{equation}

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\[(38) \quad P(\max\{|S_{(i,j)}| : 1 \leq i \leq m, 1 \leq j \leq n\}) \leq \left(3^{\frac{3}{4}} \left(2^{-\frac{1}{4}} \left(\frac{ES^{2}_{(m,n)}}{(\lambda_2 - \lambda_1)^2}\right)^\frac{3}{4}\right) (P(|S_{(m,n)}| \geq \lambda_1))^\frac{1}{4}. \]

**Proof.** We will use the ideas in the proof of Theorem 10 of Newman and Wright (1982). It follows immediately from (28), (36) and the fact that with \(\overline{S}_j\) as defined in (3.1), \(S^*_{{(m,n)}} = \overline{S}_n^*,\) that for \(0 \leq \lambda < \lambda_2,\)

\[(39) \quad P(S^2_{{(m,n)}} \geq \lambda_2) \leq \left(\frac{E(\overline{S}_n^2)}{(\lambda_2 - \lambda)^2}\right)^\frac{1}{4} (P(\overline{S}_n \geq \lambda))^\frac{1}{4}. \]

Let \(T_k = S_{(k,n)}\). Then \(T_k = X_1 + \cdots + X_k\) where \(X_i = X_{(i,1)} + \cdots + X_{(i,n)}\) so that the \(X_i\)'s are LPQD with \(EX_i = 0\). Since \(\overline{S}_n = T^*_m\) and \(S_{{(m,n)}} = T_m\), it follows from (17) of Corollary 2.4

\[(40) \quad E(\overline{S}_n^2) = E(T^*_m) \geq E(T_m^2) = E(S^2_{{(m,n)}}) \]

and from (23) of Corollary 2.5 that for \(0 \leq \lambda_1 < \lambda,\)

\[(41) \quad P(\overline{S}_n \geq \lambda) \leq \left(\frac{E(S^2_{{(m,n)}})}{(\lambda - \lambda_1)^2}\right)^\frac{1}{4} (P(S_{{(m,n)}} \geq \lambda_1))^\frac{1}{4}. \]

Combining (39), (40), and (41), we obtain

\[(42) \quad P(S^*_{{(m,n)}} \geq \lambda_2) \leq \frac{[E(S^2_{{(m,n)}})]^{\frac{3}{4}}}{(\lambda_2 - \lambda)(\lambda - \lambda_1)^\frac{1}{4}} (P(S_{{(m,n)}} \geq \lambda_1))^\frac{1}{4} \]

choosing \(\lambda = \frac{(2\lambda_1 + \lambda_2)}{3}\) to minimize the right hand side of (42) leads to (37). To obtain (38), we add to (37) the analogous inequality obtained when all the \(X_{(i,j)}\)'s are replaced by their negatives, and use the fact that for \(u, v \geq 0,\)

\[u^\frac{1}{4} + v^\frac{1}{4} \leq 2^{\frac{3}{4}} (u + v)^\frac{1}{4}. \]

\(\square\)

**Remark 2.** Theorem 3.4 is an extension of Theorem 10 of Newman and Wright (1982) for 2-parameter array of associated random variables to the LPQD case.
4. A weak convergence of 2-parameter arrays

The next theorem gives two-parameter weak convergence as a consequence of Theorem A and Theorem 3.4. We choose to consider weak convergence in the sense of [14] for the sake of convenience.

**Theorem 4.1.** Let \( \{X_j : j \geq 1\} \) be a stationary 2-dimensional array of LPQD random variables with \( EX_j = 0 \) and satisfy (3) in Theorem A for \( d = 2 \). Let \( W_n(t), W(t) \) be as in Theorem A with \( d = 2 \) and with \( (t_1, t_2) \in [0, 1]^2 \). Then \( W_n(t_1, t_2) \) converges weakly to \( W(t_1, t_2) \).

**Proof.** As in the proof of Theorem 11 in Newman and Wright [13] by Theorem A and [14, Theorem 2] it can be proved that

\[
\forall \varepsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} \delta^{-2} P(\omega(W_n, \delta) > \varepsilon) = 0,
\]

where \( \omega(W_n, \delta) = \sup\{|W_n(t) - W_n(s)| : s, t \in [0, 1]^2, |s - t| < \delta\} \) and \(|s - t| = \max(|s_1 - t_1|, |s_2 - t_2|)\). For the sake of completeness we repeat it here. Simple estimates show that to obtain (43) it suffices to have

\[
\forall \varepsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} \delta^{-2} P(\overline{\omega}(W_n, \delta) \geq \varepsilon) = 0,
\]

where

\[
\overline{\omega}(W_n, \delta) = \sup\{|W_n(t)| : t \in [0, \delta]^2\} = n^{-1} \max\{|S_{(k,j)}| : 1 \leq k \leq n\delta, 1 \leq j \leq n\delta\}.
\]

Now from the fact that \( E(S^2_{([n\delta], [n\delta])}/n^2) \to \sigma^2 \delta^2 \) (See [10]), and Theorem A we have that by putting \( \lambda_1 = n\varepsilon \) and \( \lambda_2 = n\varepsilon/2 \) in (38) of Theorem 3.4

\[
\limsup_{n \to \infty} P(\overline{\omega}(W_n, \delta) \leq (3^{1/2})(2^{-1/4}) \left(\frac{2^2 \delta^2 S^2_{([n\delta], [n\delta])}}{n^2 \varepsilon^2}\right)^{1/3} \left[ \lim_{n \to \infty} \left( P\left(\frac{S_{([n\delta], [n\delta])}}{n} \geq \frac{\varepsilon}{2}\right) \right)^{1/4} \right]
\]

\[
\leq C \left(\frac{\sigma^2 \delta^2}{\varepsilon^2}\right)^{1/4} \left[ \lim_{n \to \infty} \left( P\left(W_n(\delta, \delta) \geq \frac{\varepsilon}{2}\right) \right)^{1/4} \right]
\]

\[
\leq \left(\frac{\sigma^2 \delta^2}{\varepsilon^2}\right)^{1/4} \left[ \int_{\varepsilon/2}^{\infty} \left(2\pi \sigma^2 \delta^2\right)^{-1/2} \exp\left(-\frac{u^2}{2\sigma^2 \delta^2}\right) \, du \right]^{1/4}
\]

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where $C$ is a universal constant. Thus for fixed $\sigma$ and $\varepsilon$, we have for some constants $a$ and $b$

$$
\lim_{\delta \to 0} \lim_{n \to \infty} \sup \delta^{-2} P(\bar{\omega}(W_n, \delta) \geq \varepsilon) \leq \lim_{\delta \to 0} a \delta^{-\frac{1}{2}} \left( \int_{b \delta^{-1}}^\infty (2\pi)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} du \right)^{\frac{1}{4}} = 0
$$

which yields (44) and completes the proof. \qed

**Remark 3.** Theorem 4.1 is an extension of Theorem 11 of Newman and Wright(1982) for 2-parameter associated sequence to the $LPQD$ case.

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**References**


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