Bull. Korean Math. Soc. **35** (1998), No. 2, pp. 301–309

A LIOUVILLE-TYPE THEOREM FOR COMPLETE RIEMANNIAN MANIFOLDS

SOON MEEN CHOI, JUNG-HWAN KWON AND YOUNG JIN SUH

ABSTRACT. The purpose of this paper is to give a theorem of Liouville-type for complete Riemannian manifolds as an extension of the Theorem of Nishikawa [6].

1. Introduction

First we consider the most popular Maximum Principle. Let U be an open connected set in an m-dimensional Euclidean space $\mathbb{R}^m$ and $\{x^j\}$ a Euclidean coordinate. We denote by L a differential operator defined by

$$L = \sum a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum b^j \frac{\partial}{\partial x^j},$$

where a^{ij} and b^j are smooth functions on U for any indices. When the matrix (a^{ij}) is positive definite and symmetric, it is called *a second order elliptic differential operator.* We assume that L is an elliptic differential operator. The Maximum Principle is explained as follows:

MAXIMUM PRINCIPLE.
For a smooth function f on U if it satisfies

$$Lf \geqq 0$$

Received October 17, 1997.
1991 Mathematics Subject Classification: Primary 53C40, Secondary 53C15.
Key words and phrases: Liouville-type theorem, complete Riemannian manifolds, maximum principle, generalized maximum principle.
This paper was supported by the grants from Basic Science Research Program, BSRI 97-1404, Ministry of Education, Korea, 1997 and TGRC-KOSEF.

and if there exists a point in U at which it attains the maximum, namely, if there exists a point x_0 in U at which $f(x_0) \geqq f(x)$ for any point x in M, then the function f is constant.

In Riemannian Geometry, this property is reformed as follows. Let (M, g) be a Riemannian manifold with the Riemannian metric g. Then we denote by $\triangle$ the Laplacian associated with the Riemannian metric g. A function f is said to be *subharmonic* or *harmonic* if it satisfies

$$\triangle f \geqq 0 \quad \text{or} \quad \triangle f = 0.$$

The maximum principle on Riemannian manifolds is as follows:

MAXIMUM PRINCIPLE.
For a subharmonic function f on a Riemannian manifold M if there exists a point in M at which it attains the maximum, then the function f is constant.

This property is to give a certain condition for a subharmonic function to be constant. When we give attention to the fact relative to these Maximum Principles, we see the classical theorem of Liouville.

LIOUVILLE'S THEOREM.

(1) *Let f be a subharmonic function on $\mathbb{R}^2$. If it is bounded, then it is constant.*
(2) *Let f be a harmonic function on $\mathbb{R}^m (m \geqq 3)$. If it is bounded, then it is constant.*

We are interested in Riemannian analogues of Liouville's theorem. Compared with these last two theorems we give attention to the fact that there is an essential difference between base manifolds. In fact, one is compact and the other is complete and noncompact. We consider here a family of Riemannian manifolds $\{(M, g)\}$. At the global situation it suffices to consider about the family of complete Riemannian manifolds. Of course, the subclass of compact Riemannian manifolds

$$\{(M, g) : \text{compact Riemannian manifolds}\}$$

is a subset of the family of complete Riemannian manifolds

$$\{(M, g) : \text{complete Riemannian manifolds}\},$$

since a compact Riemannian manifold is complete. However we must notice the difference between these two classes is very big in the certain sense. As is asserted by Gromov [5], we can say that almost all complete Riemannian manifolds are noncompact. So we are interested in the Riemannian analogues of Liouville's theorem. And moreover we have several essential problems for complete and noncompact Riemannian manifolds in Mathematics and in Physics. For example, in Relativity Theory one of important problems which is closely related to Riemannian Geometry is to classify codimension one space-like foliations with fibers of constant mean curvature in a 4-dimensional Minkowski space. In this situation these fibers are complete and noncompact. Thus it is interesting to consider whether or not the Maximum Principle holds on a complete and noncompact Riemannian manifold or to construct the Maximum Principle on a complete Riemannian manifold. The Maximum Principle on a complete Riemannian manifold is usually called *Generalized Maximum Principle.*

As is already stated, each of these Maximum Principles plays an important role in each branch of Mathematics. Actually *Generalized Maximum Principles* which are later introduced are also important properties similar to the Maximum Principle in a compact Riemannian manifold or more important ones than that.

In particular, a similar property on a complete Riemannian manifold was treated by Nishikawa [6], who determined space-like hypersurfaces in a Lorentz space. His Liouville-type theorem in a complete Riemannian manifold says

THEOREM A. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If a C^2-nonnegative function f satisfies*

$$\Delta f \geq 2f^2, \tag{1.1}$$

where Δ denotes the Laplacian on M, then f vanishes identically.

The purpose of this paper is to prove the following Liouville-type theorem in a complete Riemannian manifold similar to Theorem A and to give another proof of Nishikawa's theorem. In this note, the main theorem is as follows:

THEOREM. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If a C^2-nonnegative function f satisfies*

$$\Delta f \geq c_0 f^n, \tag{1.2}$$

where c_0 is any positive constant and n is any real number greater than 1, then f vanishes identically.

The present authors would like to express their sincere gratitude to the referee for his valuable comments.

2. Preliminaries

First of all, let us introduce a *Generalized Maximum Principle* due to Omori [7] and Yau [9]. This is slightly different from the original one.

THEOREM 2.1. *Let M be an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below on M. Let G be a C^2-function bounded from below on M, then for any $\epsilon > 0$ there exists a point p such that*

$$|\nabla G(p)| < \epsilon, \quad \Delta G(p) > -\epsilon \quad \textit{and} \quad \inf G + \epsilon > G(p). \tag{2.1}$$

3. Proof of the Theorem

In this section we prove the Theorem stated in the Introduction. First of all, in order to prove our Theorem, we want to verify the following Theorem 3.1. Then our Theorem is directly obtained as a corollary of this property and hence Nishikawa's Theorem [6] is also a direct consequence of this theorem, which means that it gives another proof of Nishikawa's one.

THEOREM 3.1. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be any formula of the variable f with constant coefficients such that*

$$F(f) = c_0 f^n + c_1 f^{n-1} + \cdots + c_k f^{n-k} + c_{k+1},$$

where $n > 1$, $1 \geq n-k \geq 0$ and $c_0 > c_{k+1}$. If a C^2-nonnegative function f satisfies

$$\triangle f \geq F(f), \tag{3.1}$$

then we have

$$F(f_1) \leq 0, \tag{3.2}$$

where f_1 denotes the supermum of the given function f.

Proof. From the assumption there exists a positive number a which satisfies

$$c_{k+1} < a^n c_0.$$

For the constant a given above the function $G(f)$ with respect to the 1-variable f is defined by $(f+a)^{\frac{1-n}{2}}$, where n is the maximal degree of the function F. Then it is easily seen that G is the C^2-function so that it is bounded from above by the positive constant $a^{\frac{1-n}{2}}$ and bounded from below by 0.

By the simple calculation we have

$$\nabla G = -\frac{n-1}{2} G^{\frac{n+1}{n-1}} \nabla f, \tag{3.3}$$

and then we get

$$\triangle G = -\frac{n+1}{2} G^{\frac{2}{n-1}} \nabla G \nabla f + \frac{1-n}{2} G^{\frac{n+1}{n-1}} \triangle f,$$

hence we get by using the above equation (3.3)

$$\frac{1-n}{2} G^{\frac{2n}{n-1}} \triangle f = G \triangle G - \frac{n+1}{n-1} |\nabla G|^2. \tag{3.4}$$

Since the Ricci curvature is bounded from below by the assumption and the function G defined above satisfies the condition that it is bounded from below, we can apply the Theorem 2.1 to the function G.

Given any positive number ϵ there exists a point p at which it satisfies (2.1). From (3.1) and (3.4) the following relationship at p

$$\frac{1-n}{2}G(p)^{\frac{2n}{n-1}}\triangle f(p) > -\epsilon G(p) - \frac{n+1}{n-1}\epsilon^2 \tag{3.5}$$

can be derived, where $G(p)$ denotes $G(f(p))$. Thus for any convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$ and $\epsilon \rightarrow 0$ $(m \rightarrow \infty)$, there is a point sequence $\{p_m\}$ so that the sequence $\{G(p_m)\}$ converges to $G_0 = \inf G$ by taking a subsequence, if necessary, because the sequence is bounded and therefore each term $G(p_m)$ of the sequence satisfies (2.1). From the definition of the infimum and (2.1) we have $G(p_m) \rightarrow G_0 = \inf G$ and hence $f(p_m) \rightarrow f_1 = \sup f$, according to the definition of G and the assumption $n > 1$.

On the other hand, it follows from (3.5) we have

$$\frac{1-n}{2}G(p_m)^{\frac{2n}{n-1}}\triangle f(p_m) > -\epsilon_m G(p_m) - \frac{n+1}{n-1}\epsilon_m^2 \tag{3.6}$$

and the right side of the above inequality converges to 0, because the function G is bounded. By choosing the constant a it satisfies $c_{k+1}a^{-n} < c_0$. Accordingly, there is a positive number δ such that

$$\frac{n-1}{2}c_{k+1}a^{-n} < \delta < \frac{n-1}{2}c_0,$$

where c_0 is the constant coefficient of the maximal degree of the function F. So for a given such a $\delta > 0$ we can take a sufficiently large integer m such that

$$\frac{1-n}{2}G(p_m)^{\frac{2n}{n-1}}F(f(p_m)) > -\delta,$$

where we have used the assumption (3.1) of Theorem 3.1 and (3.6). So this inequality together with the definition of $G(p_m)$ yield

$$F(f(p_m)) < \frac{2\delta}{n-1}(f+a)(p_m)^n. \tag{3.7}$$

□

ASSERTION. *The range of the solutions of the inequality with 1-variable*

$$g(x) = F(x) - \frac{2\delta}{n-1}(x+a)^n < 0$$

is bounded.

Proof. Since the constant coefficient $c_0 - \frac{2\delta}{n-1}$ of the maximal degree of the function G is positive by the definition of the constant δ, we easily see that $\lim_{x\to\infty} g(x) = \infty$. Moreover, we get

$$g(0) = c_{k+1} - \frac{2}{n-1}a^n\delta < 0,$$

by the definition of δ, which implies that there exists a root x_1 such that $g(x_1) = 0$. This implies that the supremum of the solution of $g(x) = 0$ is bounded. It means that the set of the numbers which satisfy the inequality $g(x) < 0$ is bounded. Namely, we have our assertion. □

The inequality (3.7) and the above Assertion show that the set $\{f(p_m)\}$ is bounded. Thus the infimum of G satisfies $G_0 \neq 0$ by the definition of the function G and hence the inequality (3.6) implies that $\limsup \triangle f(p_m) \leq 0$. This means that

$$F(f_1) \leq 0$$

by the assumption (3.2) of Theorem 3.1. Now we have completed the proof of Theorem 3.1.

Proof of the Theorem. Assume that

$$F(f) = c_0 f^n, \quad c_0 > 0$$

Then we get $F(f_1) \leq 0$. Since the function f is nonnegative by the assumption, we see that f_1 is non-negative, namely we have

$$f_1 \geq 0.$$

Hence we get

$$f_1 = \sup f = 0,$$

which means that the function f vanishes identically. This means that the proof of our Theorem is completed. □

REMARK 1. Suppose that a nonnegative function f satisfies the condition (1.2). We can directly yield

$$\nabla f^{n-1} = (n-1)f^{n-2}\nabla f,$$

$$\triangle f^{n-1} = (n-1)(n-2)f^{n-3}\nabla f \nabla f + (n-1)f^{n-2}\triangle f.$$

We define a function h by f^{n-1}. If $n \geq 2$, then it satisfies

$$\triangle h \geq (n-1)c_0 h^2.$$

Thus concerning the Theorem in the case where $n \geq 2$, the condition (1.2) is equivalent to the following

$$\triangle f \geq c_1 f^2,$$

where c_1 is a positive constant. Namely, Theorem 3.1 is only essential in the case where $1 < n < 2$.

REMARK 2. In the proof of the Theorem of Nishikawa [6], the condition $n = 2$ is essential.

REMARK 3. When $n = 1$ and the function f is bounded from above, the third author [8] has proved that the Theorem also can be established. But until now without any assumption on the function f we do not know whether or not our Theorem holds in the case $n = 1$.

References

[1] S. Y. Cheng, *Liouville theorems for harmonic maps*, Proc. Symp. Pure Math. **36** (1980), 147–151.

[2] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Proc. Symp. Pure Math. **28** (1975), 333-354.

[3] Q. M. Cheng and S. M. Choi, *Complete space-like submanifolds with parallel mean curvature vector of an indefinite space form*, Tsukuba J. Math. **17** (1993), 497–512.

[4] Q. M. Cheng and H. Nakagawa, *Totally umbilic hypersurfaces*, Hiroshima. Math. J. **20** (1990), 1–10.

[5] M. Gromov, *Filling Riemannian manifolds*, J. Diff. Geom. **18** (1983), 1-147.

[6] S. Nishikawa, *On maximal spacelike hypersurfaces in a Lorenzian manifolds*, Nagoya Math. J. **95** (1984), 117–124.
[7] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–211.
[8] Y. J. Suh, *On a Kaehler manifold whose totally real bisectional curvature is bounded from below*, Nihonkai Math. J. **5** (1994), 13–32.
[9] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201-228.

SOON MEEN CHOI, TOPOLOGY AND GEOMETRY RESEARCH CENTER, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA

JUNG-HWAN KWON, DEPARTMENT OF MATHEMATICS EDUCATION, TAEGU UNIVERSITY, TAEGU 705-714, KOREA

YOUNG JIN SUH, DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA
E-mail: yjsuh@bh.kyungpook.ac.kr