Bull. Korean Math. Soc. 35 (1998), No. 2, pp. 319–324

AN EXTENSION OF THE FUGLEDE–PUTNAM THEOREM TO p–QUASIHYPONORMAL OPERATORS

MI YOUNG LEE AND SANG HUN LEE

ABSTRACT. The equation $AX = BX$ implies $A^*X = B^*X$ when A and B are normal (Fuglede–Putnam theorem). In this paper, the hypotheses on A and B can be relaxed by using a Hilbert–Schmidt operator X: Let A be p–quasihyponormal and let B^* be invertible p–quasihyponormal such that $AX = XB$ for a Hilbert–Schmidt operator X and $\| |A^*|^{1-p} \| \cdot \| |B^{-1}|^{1-p} \| \leq 1$. Then $A^*X = XB^*$.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *normal* if $T^*T = TT^*$, *hyponormal* if $TT^* \leq T^*T$, *p–hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ for $p > 0$ (see [7]), *quasihyponormal* if $T^*(T^*T - TT^*)T \geq 0$ and *p–quasihyponormal* if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$ for $p > 0$. If $p = 1$, then T is quasihyponormal and if $p = \frac{1}{2}$, then T is semi–quasihyponormal. It is well known that a p–quasihyponormal operator is a q–quasihyponormal operator for $q \leq p$. But the converse is not true in general (see [1], [7] and [8]).

The familiar Fuglede–Putnam theorem is as follows (see [4] and [6]):

THEOREM A. *If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

S. K. Berberian[2] relaxes the hypotheses on A and B in Theorem A at the cost of requiring X to be Hilbert–Schmidt class. Recently, H. K.

Received June 25, 1997. Revised November 5, 1997.
1991 Mathematics Subject Classification: 47B20.
Key words and phrases: p-hyponormal, p-quasihyponormal, Hilbert–Schmidt operator.
This paper was supported in part by BSRI-97-1401 and TGRC-KOSEF.

Cha[3] showed that the hyponormality in the result of Berberian[2] can replaced by the quasihyponormality of A and B^* under some additional conditions.

In this paper, we will introduce p–hyponormal operators and show that the quasihyponormality of A and B^* can be replaced by the p–quasihyponormality of A and B^*.

2. Main Results

In this paper, let $0 < p < 1$. Without loss of generality, we may assume that $p = 2^{-n}$ for some integer $n \geq 1$.

Let T be an operator in $\mathcal{L}(\mathcal{H})$ and let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. We define the Hilbert–Schmidt norm of T to be

$$||T||_2 = (\sum_{n=1}^{\infty} ||Te_n||^2)^{\frac{1}{2}}.$$

This definition is independent of the choice of basis (see [4]). If $||T||_2 < \infty$, then T is said to be a Hilbert–Schmidt operator and we denote the set of all Hilbert–Schmidt operator on $\mathcal{H}$ by $\mathcal{B}_2(\mathcal{H})$. Then we have the following:

THEOREM 1 [4].

(1) *The set $\mathcal{B}_2(\mathcal{H})$ is a self–adjoint ideal of $\mathcal{L}(\mathcal{H})$.*

(2) *If $(A, B) = \sum_{i=1}^{\infty}(Ae_i, Be_i) = tr(B^*A) = tr(AB^*)$ for A and B in $\mathcal{B}_2(\mathcal{H})$, then $(\cdot,\cdot)$ is an inner product on $\mathcal{B}_2(\mathcal{H})$and $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space with respect to this inner product, where $\{e_i\}$ is any orthonormal basis for $\mathcal{H}$ and $tr(\cdot)$ denotes the trace.*

For each pair of operators A and B in $\mathcal{L}(\mathcal{H})$, an operator $\mathcal{J}$ in $\mathcal{L}(\mathcal{B}_2(\mathcal{H}))$ is defined by

$$\mathcal{J}X = AXB.$$

Evidently $||\mathcal{J}|| \leq ||A|| \cdot ||B||$. And the adjoint of $\mathcal{J}$ is given by the formula $\mathcal{J}^*X = A^*XB^*$(more precisely, see [2]). In particular, if A and B are both positive, then $\mathcal{J}$ is positive and $\mathcal{J}^{\frac{1}{2}}X = A^{\frac{1}{2}}XB^{\frac{1}{2}}$, as one sees from the calculation

$$\begin{aligned}(\mathcal{J}X, X) &= tr(AXBX^*) = tr(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}})\\ &= tr((A^{\frac{1}{2}}XB^{\frac{1}{2}})(A^{\frac{1}{2}}XB^{\frac{1}{2}})^*) \geq 0.\end{aligned}$$

Since $|\mathcal{J}|^2X = |A|^2X|B^*|^2$ and $|\mathcal{J}^*|^2X = |A^*|^2X|B|^2$, we have

$$|\mathcal{J}|^{\frac{1}{2^n}}X = |A|^{\frac{1}{2^n}}X|B^*|^{\frac{1}{2^n}}$$

and

$$|\mathcal{J}^*|^{\frac{1}{2^n}}X = |A^*|^{\frac{1}{2^n}}X|B|^{\frac{1}{2^n}}$$

for each integer $n \geq 1$.

Now, we need the following lemmas.

LEMMA 2. *If A and B^* are p–quasihyponormal, then the operator $\mathcal{J}$ on $\mathcal{B}_2(\mathcal{H})$ defined by $\mathcal{J}X = AXB$ is also p–quasihyponormal.*

Proof. For $X \in \mathcal{B}_2(\mathcal{H})$, we have

$$\begin{aligned}\mathcal{J}^*(|\mathcal{J}|^{2p} - |\mathcal{J}^*|^{2p})\mathcal{J}X &= \mathcal{J}^*(|\mathcal{J}|^{2p} - |\mathcal{J}^*|^{2p})AXB\\ &= A^*|A|^{2p}AXB|B^*|^{2p}B^* - A^*|A^*|^{2p}AXB|B|^{2p}B^*\\ &= A^*(|A|^{2p} - |A^*|^{2p})AXB|B^*|^{2p}B^*\\ &+ A^*|A|^{2p}AXB(|B^*|^{2p} - |B|^{2p})B^*.\end{aligned}$$

Since A and B^* are p–quasihyponormal, we have

$$\mathcal{J}^*(|\mathcal{J}|^{2p} - |\mathcal{J}^*|^{2p})\mathcal{J} \geq 0. \qquad \square$$

LEMMA 3.

(1) *If T is invertible p–hyponormal, T^{-1} is also p–hyponormal.*
(2) *Let T be invertible. Then T is p–hyponormal if and only if T is p–quasihyponormal.*

Proof. Note that if T is invertible, then $|T|$ is invertible.
(1) Since $(T^*T)^p - (TT^*)^p \geq 0$, we have

$$(T^*T)^{-\frac{p}{2}}((T^*T)^p - (TT^*)^p)(T^*T)^{-\frac{p}{2}} \geq 0.$$

This is equivalent to

$$I \geq (T^*T)^{-\frac{p}{2}}(TT^*)^p(T^*T)^{-\frac{p}{2}}.$$

It is well known that $A \geq I$ implies $A^{-1} \leq I$. Thus

$$\begin{aligned}0 &\leq (T^*T)^{\frac{p}{2}}(TT^*)^{-p})(T^*T)^{\frac{p}{2}} - I \\ &= (T^*T)^{\frac{p}{2}}((TT^*)^{-p} - (T^*T)^{-p})(T^*T)^{\frac{p}{2}}.\end{aligned}$$

This is equivalent to

$$\begin{aligned}0 &\leq (TT^*)^{-p} - (T^*T)^{-p} \\ &= ((T^{-1})^*T^{-1})^p - (T^{-1}(T^{-1})^*)^p.\end{aligned}$$

So, T^{-1} is p–hyponormal.
(2) Suppose that T is p–quasihyponormal. Then we have

$$\begin{aligned}0 &\leq (T^{-1})^*(T^*((T^*T)^p - (TT^*)^p)T)T^{-1} \\ &= (T^*T)^p - (TT^*)^p.\end{aligned}$$

The converse is trivial by the definition. □

The following inequality due to McCarthy is an operator variant of the Hölder Inequality (see [5] and [9]).

HÖLDER-MCCARTHY INEQUALITY. *Let A be a positive operator on $\mathcal{H}$. Then the following inequalities hold:*

(1) $(A^rx, x) \leq ||x||^{2(1-r)}(Ax, x)^r$ *for* $x \in \mathcal{H}$ *if* $0 < r \leq 1$.
(2) $(A^rx, x) \geq ||x||^{2(1-r)}(Ax, x)^r$ *for* $x \in \mathcal{H}$ *if* $r \geq 1$.

THEOREM 4. *Let A be p–quasihyponormal and let B^* be invertible p–quasihyponormal such that $AX = XB$ for $X \in \mathcal{B}_2(\mathcal{H})$ and $\||A^*|^{1-p}\| \cdot \||B^{-1}|^{1-p}\| \leq 1$. Then $A^*X = XB^*$.*

Proof. Let $\mathcal{J}$ on $\mathcal{B}_2(\mathcal{H})$ be definded by $\mathcal{J}Y = AYB^{-1}$ for all $Y \in \mathcal{B}_2(\mathcal{H})$. Since $(B^*)^{-1} = (B^{-1})^*$ is p–quasihyponormal by Lemma 3, Lemma 2 implies that $\mathcal{J}$ is p–quasihyponormal. Since $\mathcal{J}X = X$ and since $\mathcal{J}$ is p–quasihyponormal, we have

$$((\mathcal{J}^*\mathcal{J})^pX, X) \geq ((\mathcal{J}\mathcal{J}^*)^pX, X).$$

By Hölder–McCarthy inequality, we have

$$\begin{aligned}\||\mathcal{J}^*|^pX\|^2 &\leq ((\mathcal{J}^*\mathcal{J})^pX, X) \\ &\leq \|X\|^{2(1-p)}(\mathcal{J}^*\mathcal{J}X, X)^p = \|X\|^2,\end{aligned}$$

and hence

$$\begin{aligned}\|\mathcal{J}^*X\| &\leq \||\mathcal{J}^*|^{1-p}\| \cdot \||\mathcal{J}^*|^pX\| \\ &\leq \||A^*|^{1-p}\| \cdot \||B^{-1}|^{1-p}\| \cdot \|X\| \leq \|X\|.\end{aligned}$$

Thus $\|\mathcal{J}^*X - X\|^2 \leq 0$. So, $A^*X(B^{-1})^* = X$ and the proof is complete. □

As consequences of Theorem 4, we obtain

COROLLARY 5 [3, THEOREM 3]. *Let A be quasihyponormal and let B^* be invertible quasihyponormal such that $AX = XB$ for $X \in \mathcal{B}_2(\mathcal{H})$. Then $A^*X = XB^*$.*

COROLLARY 6 [2, THEOREM]. *Suppose A, B and X are operators in the Hilbert space $\mathcal{H}$, such that $AX = XB$. Assume also that X is an operator of Hilbert–Schmidt class. Then $A^*X = XB^*$ under either of the following hypotheses:*

(1) *A and B^* are hyponormal;*
(2) *B is invertible and $\|A\| \cdot \|B^{-1}\| \leq 1$.*

References

[1] S. C. Arora and P. Arora, *On p–quasihyponormal operators for* $0 < p < 1$, Yokohama Math. J. **41** (1993), 25–29.
[2] S. K. Berberian, *Extensions of a theorem of Fuglede and Putnam*, Proc. Amer. Math. Soc. **71** (1978), 113–114.
[3] H. K. Cha, *An extensions of a theorem of Fuglede–Putnam theorem to quasi–hyponormal operators using a Hilbert-schmidt operator*, Youngnam Math. J. **1** (1994), 73–76.
[4] J. B. Conway, *Subnormal operators*, vol. 51, Research Notes in Math., Pitman Advanced Pub. Program, 1981.
[5] M. Fujii, R. Nakamoto and H. Watanabe, *The Heinz-Kato-Furuta inequality and hyponormal operators*, Math. Japonica **40** (1994), 469–472.
[6] P. R. Halmos, *A Hilbert space problem book*, Springer–Verlag, New York, 1974.
[7] M. Y. Lee and S. H. Lee, *Some remarks on the structure of p–hyponormal operators*, submit.
[8] M. Y. Lee and S. H. Lee, *Some generalized theorems on p–quasihyponormal operators for* $0 < p < 1$, to appear in Nihonkai Math. J.
[9] C. A. McCarthy, C_ρ, Israel J. Math. **5** (1967), 249–271.
[10] I. H. Sheth, *Quasihyponormal operators*, Rev. Roumaine Math. Pures Appl. **19** (1976), 1049–1053.

Department of Mathematics, College of Natural Science, Kyungpook National University, Taegu 702–701, Korea