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ERROR ESTIMATES FOR A FREQUENCY–DOMAIN FINITE ELEMENT METHOD FOR PARABOLIC PROBLEMS WITH A NEUMANN BOUNDARY CONDITION

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ABSTRACT. We introduce and analyze a naturally parallelizable frequency–domain method for parabolic problems with a Neumann boundary condition. After taking the Fourier transformation of given equations in the space–time domain into the space–frequency domain, we solve an indefinite, complex elliptic problem for each frequency. Fourier inversion will then recover the solution of the original problem in the space-time domain. Existence and uniqueness of a solution of the transformed problem corresponding to each frequency is established. Fourier invertibility of the solution in the frequency–domain is also examined. Error estimates for a finite element approximation to solutions of transformed problems and full error estimates for solving the given problem using a discrete Fourier inverse transform are given.

1. Introduction

Let Ω be the unit open cube in $\mathbb{R}^d$, $d = 2, 3$, $J = [0, T]$, $T > 0$, and $\Gamma = \partial\Omega$. We are interested in a numerical method for the following parabolic model problem:

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$$\lambda \frac{\partial u}{\partial t}(x,t) - \nabla \cdot (\kappa \nabla u)(x,t) = f(x,t) \quad \text{in } \Omega \times J, \tag{1.1}$$

$$\frac{\partial u}{\partial n}(x,t) = 0 \quad \text{on } \Gamma \times J, \tag{1.2}$$

$$u(x,0) = 0 \quad \text{for } x \in \Omega, \tag{1.3}$$

where $n = (n_{x_1}, n_{x_2}, \cdots, n_{x_d})$ is the unit outward normal vector on Γ, and $\lambda \in L^2(\Omega)$, $\kappa \in W^{1,\infty}(\Omega)$ are positive functions of x defined on Ω, which satisfy $0 < \kappa_* \le \kappa \le \kappa^*$, $|\nabla \kappa| \le \kappa^*$ and $0 < \lambda_* \le \lambda \le \lambda^*$ where $\kappa_*, \kappa^*, \lambda_*$, and λ^* are constants.

Problem (1.1) often describes the temperature distribution of an insulated isotropic inhomogeneous medium with the heat capacity $\lambda(x)$, $x \in \Omega$ and the thermal conductivity $\kappa(x)$, $x \in \Omega$ subject to a time-limited heat source $f(x,t) \in \Omega \times J$ and the initial temperature distribution $u(x,0) = 0$, $x \in \Omega$.

The most popular effective methods to get a numerical solution of (1.1) are to approximate the solution of the problem in the space-time domain by using a marching algorithm such as backward-Euler or Crank-Nicholson methods. Such methods have proven to be applicable to many practical problems. In order to advance to next time steps when one uses a marching algorithm, one needs to solve elliptic problems using informations on space grids at the current and/or previous time steps. It is also well-known that shorter time steps are needed when one wishes to capture sharper initial changes near $t = 0$.

In this paper, we propose and analyze an alternative (finite element) numerical method, frequency–domain method, to approximate the solution of Problem (1.1) by using the Fourier transformations.

We first transform the problem (1.1) into a set of elliptic problems for discrete number of different frequencies of interest by taking the Fourier transformation in time t. We then approximate the solution of the transformed elliptic problem corresponding to each frequency. The numerical solution of Problem (1.1) at a given time is then recovered by a discrete Fourier inverse transform.

The main characteristics of the frequency–domain formulation is that the elliptic problem corresponding to one frequency is completely independent of the other problems corresponding to the other frequencies. Therefore we are able to solve the set of elliptic problems simultaneously by assigning problems with different frequencies to different processors, if there are lots of processors available. Then our numerical solutions at any given time t of Problem (1.1) is recovered by combining the solutions using a discrete inverse Fourier transform. Independence of each problem guarantees no communication cost among processors. Thus the frequency–domain formulation for Problem (1.1) may give us a very natural parallel algorithm.

On the other hand, in the recent decade, there have been remarkable advances on parallelization with respect to the spatial discretization; for instance methods based upon decomposing the domain into subdomains using the idea of domain decomposition methods in solving each elliptic problem corresponding to each fixed time step. For some accounts for such methods, see some of [4, 5, 12, 13, 15, 16, 22] and recent publications in major numerical analysis journals. However, these methods require heavy communication cost among processors in order to pass informations between neighboring subdomains. In this sense, parallel algorithms based on the space–time formulation are not naturally parallelizable. Thus we might say that the most favorable advantage for our scheme lies in the natural parallelization when massively parallel processors are available.

The above frequency–domain procedure has been proven to be very efficient for solving wave propagations with absorbing boundary conditions in a parallel machine [9, 10]. Wave equations becomes Helmholtz-type equations in the space-frequency domain, which have eigensolutions with Dirichlet or Neumann boundary conditions. This is not the case with absorbing boundary conditions; with such conditions the Helmholtz-type equations are uniquely solvable, and thus a natural parallelization is possible in the sense mentioned above. for details, see [9, 10, 11, 17]. Recently frequency–domain approaches to parabolic problems with other kinds of boundary conditions were analyzed, see [19, 23]. See also [18] for an analysis of a linearized Navier-Stokes equations, where a similar treatment for the Dirichlet boundary condition has been done.

This paper is organized as follows. In §2, we give the frequency–domain formulation for Problem (1.1) and show that the problem in the frequency–domain has the unique solution for each positive frequency, and regularity and stability results are given for such solutions. In §3 we first treat a finite element procedure for a transformed problem corresponding to a single frequency and derive error estimates for the procedure. We then give full error estimate for solving (1.1) via the inverse Fourier transformation.

2. Frequency–Domain Approach

2.1. Problems in the Frequency–Domain

Recall first that the Fourier transform $\widehat{v}(\cdot,\omega)$ of a function $v(\cdot,t)$ in time is defined by

$$\widehat{v}(\cdot,\omega) = \int_{-\infty}^{\infty} v(\cdot,t)e^{-i\omega t}dt$$

and the Fourier inversion formula given by

$$v(\cdot,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{v}(\cdot,\omega)e^{i\omega t}d\omega.$$

Note also that if $v(x,t)$ is a real function, its Fourier transform satisfies the conjugate relation:

$$\widehat{v}(x,-\omega) = \overline{\widehat{v}(x,\omega)}, \quad \omega \in R. \tag{2.1}$$

Then, the Fourier inversion formula takes the form

$$v(x,t) = \frac{1}{\pi}\mathrm{Re}\int_{0}^{\infty} \widehat{v}(x,\omega)e^{i\omega t}\, d\omega. \tag{2.2}$$

In order to take the Fourier transform of Problem (1.1), we shall assume that the source function $f(x,t)$ in (1.1) is defined on $\Omega \times (0,\infty)$. We then extend u and f by zero for $t < 0$ and transform the space–time formulation of the equations (1.1) to a space–frequency formulation by taking the Fourier transform of (1.1) with respect to the time variable t. We then obtain a set of the following elliptic problems:

For each ω, find $\widehat{u}(x,\omega)$ such that

$$i\omega\lambda\widehat{u} - \nabla\cdot(\kappa\nabla\widehat{u}) = \widehat{f}, \quad x \in \Omega, \tag{2.3}$$

$$\frac{\partial\widehat{u}}{\partial n} = 0, \quad x \in \Gamma. \tag{2.4}$$

Since the source $f(x,t)$ is real, an application of (2.1) to the equations (2.3) leads to $\widehat{u}(x,-\omega) = \overline{\widehat{u}(x,\omega)}$. Therefore it suffices to find solution

$\widehat{u}(x,\omega)$ of Problem (2.3) for all $\omega \geq 0$, and then the solution $u(x,t)$ of Problem (1.1) is found by using the Fourier inversion formula.

If $\omega = 0$, then (2.3) becomes a Poisson's equation with a Neumann boundary condition; a solution exists and is unique up to an additive constant so long as

$$\int_\Omega \widehat{f}(x,0)dx = 0. \tag{2.5}$$

It then follows from (2.5) that the mean value of the source $f(x,t)$ must satisfy the following condition;

$$\int_\Omega \int_0^\infty f(x,t)dt\, dx = 0.$$

REMARK 2.1. If one chooses an appropriate quadrature, e.g., the mid-point rule, for a discrete Fourier inversion formula, one needs not to consider the problem (2.3) corresponding to $\omega = 0$. Henceforth we will confine ourselves to the cases $\omega > 0$.

2.2. Variational Formulation

All functions are assumed to have values in the complex field $\mathbb{C}$. But, they are considered in the real field for the time-dependent problems. Standard notations for function spaces and their norms will be used in this paper. See [1, 8] for more details of function spaces and related norms.

For each given $\omega > 0$, define the sesquilinear form $a_\omega(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$a_\omega(u,v) = i\omega\,(\lambda u, v) + (\kappa\nabla u, \nabla v), \qquad u, v \in H^1(\Omega).$$

A *variational formulation* of problem (2.3) is then as follows;

For each frequency $\omega > 0$, find $\widehat{u}(\cdot,\omega) \in H^1(\Omega)$ such that for a given $\widehat{f} \in L^2(\Omega)$

$$a_\omega(\widehat{u}, v) = (\widehat{f}, v), \quad v \in H^1(\Omega). \tag{2.6}$$

An application of the Green's theorem immediately gives that *if $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$ for each frequency $\omega > 0$, then $\widehat{u}(\cdot,\omega) \in H^2(\Omega)$ is a solution of (2.6) if and only if $\widehat{u}(\cdot,\omega)$ is a solution of (2.3).*

2.3. Uniqueness and Existence

In what follows, C will denote a generic positive constant which may differ from place to place.

THEOREM 2.1. *Let $\omega > 0$ be given. Assume that $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$. Then the equation* (2.6) *has a unique solution $\widehat{u}(\cdot,\omega) \in H^1(\Omega)$.*

Proof. From the definition of $a_\omega(\cdot,\cdot)$,

$$\begin{aligned} |a_\omega(\widehat{u},\widehat{u})| &= \left[\omega^2\|\sqrt{\lambda}\widehat{u}\|^4 + \|\sqrt{\kappa}\nabla\widehat{u}\|^4\right]^{1/2} \geq \frac{1}{\sqrt{2}}\left[(\omega\lambda_*)\|\widehat{u}\|^2 + \kappa_*\|\nabla\widehat{u}\|^2\right] \\ &\geq \frac{C}{\sqrt{2}}\min\{\omega,1\}\|\widehat{u}\|_1^2. \end{aligned}$$

Therefore we have

$$|a_\omega(\widehat{u},\widehat{u})| \geq C_\omega\|\widehat{u}\|_1^2,$$

where C_ω only depends on Ω and λ_*,κ_* and ω. Thus, $a_\omega(\cdot,\cdot)$ is *coercive.* We also have the following inequality: for $\widehat{u},\widehat{v} \in H^1(\Omega)$

$$\begin{aligned} |a_\omega(\widehat{u},\widehat{v})| &\leq \omega\left\|\sqrt{\lambda}\widehat{u}\right\|\left\|\sqrt{\lambda}\widehat{v}\right\| + \|\sqrt{\kappa}\nabla\widehat{u}\|\|\sqrt{\kappa}\nabla\widehat{v}\| \\ &\leq \omega\lambda^*\|\widehat{u}\|\,\|\widehat{v}\| + \kappa^*\|\nabla\widehat{u}\|\|\nabla\widehat{v}\|. \end{aligned}$$

Therefore we have

$$|a_\omega(\widehat{u},\widehat{v})| \leq C(1+\omega)\|\widehat{u}\|_1\|\widehat{v}\|_1,$$

where C only depends on Ω, κ^* and λ^*. Thus, $a_\omega(\cdot,\cdot)$ is *continuous.* An application of the Lax-Milgram lemma [8, 21] gives uniqueness and existence of the solution of Problem (2.6). □

2.4. Regularity and Stability

We are now going to establish stability and regularity of the solution of the problem (2.3). For given $\omega > 0$, we begin with taking $v = \widehat{u}$ in (2.6):

$$i\omega(\lambda\widehat{u},\widehat{u}) + (\kappa\nabla\widehat{u},\nabla\widehat{u}) = (\widehat{f},\widehat{u}). \tag{2.7}$$

The imaginary part of (2.7) gives

$$\omega\lambda_*\|\widehat{u}\|^2 \leq \omega\left\|\sqrt{\lambda}\widehat{u}\right\|^2 = \mathrm{Im}\,(\widehat{f},\widehat{u}) \leq \|\widehat{f}\|\|\widehat{u}\|.$$

Therefore we have

$$\|\widehat{u}\| \leq C\frac{1}{\omega}\|\widehat{f}\|. \tag{2.8}$$

On the other hand, the real part of (2.7) gives

$$\kappa_*\|\nabla\widehat{u}\|^2 \;\leq\; \|\sqrt{\kappa}\nabla\widehat{u}\|^2 = \mathrm{Re}\,(\widehat{f},\widehat{u}) \leq \|\widehat{f}\|\|\widehat{u}\|.$$

From (2.8) it follows that

$$\|\nabla\widehat{u}\| \leq C\frac{1}{\sqrt{\omega}}\|\widehat{f}\|.$$

Summarizing the above estimates, one gets the following lemma.

LEMMA 2.1. *Let $\widehat{u}(\cdot,\omega) \in H^1(\Omega)$ be a solution of Equation* (2.6) *for a given $\omega > 0$. Then we have the following estimates:*

$$\|\widehat{u}(\cdot,\omega)\| \;\leq\; C\frac{1}{\omega}\|\widehat{f}(\cdot,\omega)\|, \tag{2.9}$$

$$|\widehat{u}(\cdot,\omega)|_1 \;\leq\; C\frac{1}{\sqrt{\omega}}\|\widehat{f}(\cdot,\omega)\|, \tag{2.10}$$

Let us now turn to an $H^2(\Omega)$-estimate for the solution $\widehat{u}$ of Problem (2.3). First, we need the following result. See [11] or [23] for a proof.

LEMMA 2.2. *If $\widehat{u}(\cdot,\omega) \in H^2(\Omega)$ is a solution of Problem* (2.3), *then the following estimate holds.*

$$\sum_{i,j=1}^{d}\left\|\frac{\partial^2\widehat{u}}{\partial x_i\partial x_j}\right\|^2 \leq C\|\Delta\widehat{u}\|^2. \tag{2.11}$$

Using the above lemmas and the condition which $\kappa \in W^{1,\infty}(\Omega)$, we now have the following result.

LEMMA 2.3. *Assume that $\omega > 0$ is given and $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$. If $\widehat{u}(\cdot,\omega) \in H^2(\Omega)$ be the solution of* (2.6), *then there exists a positive constant C such that*

$$|\widehat{u}(\cdot,\omega)|_2 \leq C(1+\omega^{-1/2})\|\widehat{f}(\cdot,\omega)\|.$$

In particular, the estimate of Lemma 2.3 shows the existence of $\widehat{u} \in H^2(\Omega)$ if $\widehat{f} \in L^2(\Omega)$ by the method of Galerkin approximation [20].

We summarize the above results in the following theorem.

THEOREM 2.2. *For any $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$, there exists a unique solution $\widehat{u}(\cdot,\omega) \in H^2(\Omega)$ with*

$$\|\widehat{u}(\cdot,\omega)\|_2 \le C(1+\omega^{-1})\|\widehat{f}(\cdot,\omega)\|.$$

As an immediate result of Theorem 2.2, we have the following.

COROLLARY 2.1. *If $\|(1+\omega^{-1})\widehat{f}(\cdot,\omega)\|$ is integrable with respect to ω over the frequency domain $\mathbb{R}$, then there exist a Fourier inverse $u(x,t)$ of the solution $\widehat{u}(\cdot,\omega) \in H^2(\Omega)$.*

3. Finite Element Approximation and Error Estimates

Here we first derive error estimates for a finite element procedure for a transformed problem corresponding to a single frequency and then give full error estimates for approximating the solution of (1.1) via the inverse Fourier transformation.

3.1. Error Estimates for a Single Frequency

Let $h > 0$ be a discretization parameter tending to zero and $V_h \subset H^1(\Omega)$ be a finite element space. Then the discrete problem corresponding to (2.6) reads:

For each $\omega > 0$, find $\widehat{u} \in V_h$ such that for a given $\widehat{f} \in L^2(\Omega)$,

$$a_\omega(\widehat{u}, v) = (\widehat{f}, v), \qquad v \in V_h. \tag{3.1}$$

We shall assume that V_h satisfy the following property: There exist a positive constant C and an operator $\pi_h : H^2(\Omega) \to V_h$, independent of h such that

$$\|v - \pi_h v\|_k \le C h^{2-k} |v|_2, \quad v \in H^2(\Omega), \quad k = 0, 1. \tag{3.2}$$

For such finite element spaces, we refer, for example, [2, 3, 6, 7, 14]. Let $\widehat{u}_h(\cdot,\omega) \in V_h$ be the Galerkin approximation to $\widehat{u}(\cdot,\omega)$ of (2.6). Then $\widehat{u}_h(\cdot,\omega)$ exists uniquely due to Theorem 2.1. Furthermore we have the following error estimates:

THEOREM 3.1. *Assume that $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$. Then the approximate solution $\widehat{u}_h(\cdot,\omega)$ of (3.1) to the solution $\widehat{u}(\cdot,\omega)$ of (2.6) for each frequency $\omega > 0$ satisfies that*

$$
\begin{aligned}
(3.3) \qquad \|\widehat{u}(\cdot,w) - \widehat{u}_h(\cdot,w)\|_1 &\le C(\omega + \omega^{-3/2})h\|\widehat{f}(\cdot,\omega)\|, \\
(3.4) \qquad \|\widehat{u}(\cdot,w) - \widehat{u}_h(\cdot,w)\| &\le C(\omega^2 + \omega^{-2})h^2\|\widehat{f}(\cdot,\omega)\|.
\end{aligned}
$$

Proof. From (2.6) and (3.1), we have the error equation:

$$a_\omega(\widehat{u} - \widehat{u}_h, v) = 0, \qquad v \in V_h,$$

which implies, for arbitrary $\chi \in V_h$

$$a_\omega(\widehat{u} - \widehat{u}_h, \widehat{u} - \widehat{u}_h) = a_\omega(\widehat{u} - \widehat{u}_h, \widehat{u} - \chi).$$

The real and imaginary parts of the above equation and continuity of a_ω yield the following estimates:

$$
\begin{aligned}
\kappa_*\|\nabla(\widehat{u} - \widehat{u}_h)\|^2 &\le C(1+\omega)\|\widehat{u} - \widehat{u}_h\|_1\|\widehat{u} - \chi\|_1 \\
\omega\lambda_*\|\widehat{u} - \widehat{u}_h\|^2 &\le C(1+\omega)\|\widehat{u} - \widehat{u}_h\|_1\|\widehat{u} - \chi\|_1
\end{aligned}
$$

Using the above two equations, (3.2) and Lemma 2.3, an appropriate choice of χ yields

$$
\begin{aligned}
\|\widehat{u} - \widehat{u}_h\|_1 &\le C\left(\omega + \frac{1}{\omega}\right)\|\widehat{u} - \chi\|_1 \\
&\le C\,h\left(\omega + \frac{1}{\omega}\right)|\widehat{u}|_2 \\
&\le C\,h\left(\omega + \frac{1}{\omega^{3/2}}\right)\|\widehat{f}\|,
\end{aligned}
$$

which proves (3.3).

For a proof of the second inequality the usual duality argument will be used. Let $z \in H^2(\Omega)$ be the solution of

$$a_\omega(z, v) = (\widehat{u} - \widehat{u}_h, v), \quad v \in H^1(\Omega).$$

Then, from Lemma 2.3, we have

$$|z|_2 \le C\,(1 + \omega^{-1/2})\|\widehat{u} - \widehat{u}_h\|.$$

Using the continuity of a_ω, the error equation, (3.2), (3.3), and the above estimate, we have

$$\begin{aligned}
\|\widehat{u}-\widehat{u}_h\|^2 &= a_\omega(z,\widehat{u}-\widehat{u}_h) \\
&= |a_\omega(z-\pi_h z,\widehat{u}-\widehat{u}_h)| \\
&\le C(1+\omega)\|\widehat{u}-\widehat{u}_h\|_1\,\|z-\pi_h z\|_1 \\
&\le C(1+\omega)h\|\widehat{u}-\widehat{u}_h\|_1\,|z|_2 \\
&\le C(1+\omega)(\omega+\omega^{-3/2})(1+\omega^{-1/2})h^2\|\widehat{f}\|\,\|\widehat{u}-\widehat{u}_h\|.
\end{aligned}$$

A simplification of the coefficient of the last term completes the proof. □

3.2. Full Error Estimate

We are now going to give the full estimate of errors for a fixed time t introduced by the truncation and discretization of a quadrature of the inverse Fourier transform, and caused by finite element approximations. First, for nonnegative integers $k \ge l$, let us define a function $\mathcal{P}_k^l(t)$ for a given function $f(x,t)$ on $\Omega \times (0,\infty)$ as follows.

$$\begin{aligned}
\mathcal{P}_k^0(t) &= \left(\int_0^t e^{t-\sigma}\|\sigma^k f(x,\sigma)\|^2 d\sigma\right)^{\frac{1}{2}}, \\
\mathcal{P}_k^l(t) &= \left(\int_0^t\int_0^{\sigma_l}\cdots\int_0^{\sigma_1} e^{t-\sigma}\|\sigma^{k-l} f(x,\sigma)\|^2 d\sigma d\sigma_1\cdots d\sigma_l\right)^{\frac{1}{2}} \quad \text{for } l>0
\end{aligned}$$

LEMMA 3.1. *Let $u(x,t)$ be the solution of Problem (1.1). Suppose that for a given integer $k \ge 0$, $\mathcal{P}_k^l(t) \in L^2(0,\infty)$ for all $0 \le l \le k$. Then there is a positive constant $C = C(\lambda_*)$ such that*

$$\|t^k u(x,t)\|^2_{L^2((0,\infty);L^2(\Omega))} \le C\sum_{l=0}^{k}\|\mathcal{P}_k^l(t)\|^2_{L^2(0,\infty)}. \tag{3.5}$$

Proof. For $k=0$, multiplying (1.1) by $u(x,t)$, we have

$$\frac{1}{2}\frac{d}{dt}(\lambda u,u)+(\kappa\nabla u,\nabla u) = (f,u).$$

Thus for any $\epsilon > 0$, we have

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\left\|\sqrt{\lambda}u\right\|^2 &\le \frac{1}{4\epsilon}\|f\|^2+\epsilon\|u\|^2 \\
&\le \frac{1}{4\epsilon}\|f\|^2+\epsilon\frac{1}{\lambda_*}\left\|\sqrt{\lambda}u\right\|^2. \qquad (3.6)
\end{aligned}$$

Multiplying (3.6) by $e^{-2\epsilon bt}$, $b = \frac{1}{\lambda_*}$ and integrating it from 0 to t, we obtain

$$(3.7)\quad \lambda_* \|u(\cdot,t)\|^2 \leq \left\|\sqrt{\lambda}u(\cdot,t)\right\|^2 \leq e^{2\epsilon bt}\int_0^t \frac{e^{-2\epsilon b\sigma}}{2\epsilon}\|f(\cdot,\sigma)\|^2 d\sigma.$$

Therefore with the choice of $\epsilon = \frac{1}{2b}$,

$$(3.8)\qquad \|u(\cdot,t)\|^2 \leq C\int_0^t e^{t-\sigma}\|f(\cdot,\sigma)\|^2 d\sigma = C\{\mathcal{P}_0^0(t)\}^2.$$

Thus the inequality (3.5) follows for $k=0$.

For $k=1$, multiplying (1.1) by t, we obtain

$$\lambda(tu(\cdot,t))_t - \nabla\cdot(\kappa t\nabla u(\cdot,t)) \;=\; tf(\cdot,t) + \lambda u(\cdot,t).$$

Again multiplying the above equation by tu and following similar argument as for $k=0$, we have for any $\eta > 0$,

$$\frac{d}{dt}\frac{1}{2}\left\|\sqrt{\lambda}tu\right\|^2 \;\leq\; \frac{1}{4\eta}\left(\|tf\|^2 + \left\|\sqrt{\lambda}u\right\|^2\right) + \eta(b+1)\left\|\sqrt{\lambda}tu\right\|^2.$$

Then the same process to get (3.7), the choice of $\eta = \frac{1}{2(b+1)}$ and (3.8) give

$$\begin{aligned}\|tu(\cdot,t)\|^2 &\leq C\left(\int_0^t e^{t-\sigma}\|\sigma f(\cdot,\sigma)\|^2 d\sigma + \int_0^t\int_0^{\sigma_1} e^{t-\sigma}\|f(\cdot,\sigma)\|^2 d\sigma d\sigma_1\right)\\ &= C\left(\{\mathcal{P}_1^0(t)\}^2 + \{\mathcal{P}_1^1(t)\}^2\right).\end{aligned}$$

Thus the inequality (3.5) follows for $k=1$. Repeating the similar argument as the above for $k>1$ completes the proof. □

We also have the following lemma under an additional assumption on $f(x,t)$.

LEMMA 3.2. *Let $u(x,t)$ be the solution of Problem* (1.1). *Suppose that for a nonnegative integer m, the assumption of* Lemma 3.1 holds for $0 \leq l \leq k \leq m$ and $f(x,t) \in L^2((0,\infty); L^2(\Omega))$ in Problem (1.1) and

$$\int_0^\infty t^{2k}\int_0^t \|f(\cdot,s)\|^2\, ds\, dt < \infty,$$

$k = 0, 1, 2, \ldots, m$. Then we have the following estimates: for $k = 0, 1, 2, \ldots$

$$\text{(3.9)} \quad \|t^k u(\cdot, t)\|^2_{L^2((0,\infty);H^1(\Omega))} \leq C \left\{ \int_0^\infty t^{2k} \int_0^t \|f(\cdot, s)\|^2 \, ds \, dt + \sum_{l=0}^{k} \|\mathcal{P}_k^l(t)\|^2_{L^2(0,\infty)} \right\}.$$

Proof. Multiply (1.1) by $u_t(\cdot, t)$ to get, for any $\epsilon > 0$,

$$\lambda_* \|u_t(\cdot, t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\kappa^{1/2} \nabla u(\cdot, t)\|^2 \leq \frac{1}{4\epsilon} \|f(\cdot, t)\|^2 + \epsilon \|u_t(\cdot, t)\|^2,$$

from which we have with a choice of a sufficiently small $0 < \epsilon < \lambda_*$

$$\|u_t(\cdot, t)\|^2 + \frac{d}{dt} \|\kappa^{1/2} \nabla u(\cdot, t)\|^2 \leq C \|f(\cdot, t)\|^2.$$

By integrating the last inequality with respect to s over $[0, t]$ for any positive t, we get

$$\|\nabla u(\cdot, t)\|^2 \leq C \|\kappa^{1/2} \nabla u(\cdot, t)\|^2 \leq C \int_0^t \|f(\cdot, s)\|^2 ds.$$

Multiplying by t^{2k} both sides of the above inequality and then integrating over $(0, \infty)$ in t, and using (3.5) we obtain the desired estimate (3.9). This completes the proof. □

REMARK 3.1. Note that the integrals with respect to t over $(0, \infty)$ in Lemmas 3.1 and 3.2 can be replaced by integrals over $(0, T^*)$ for any $T^* > 0$.

We are now in a position to estimate the full errors of our algorithm when we approximate the solution $u(x, t)$ of Problem (1.1) for a given time t.

We consider restricted sources such that $|\widehat{f}(\cdot, \omega)|$ is square integrable with respect to ω over $(0, \infty)$ and thus negligible for large $|\omega|$. We then choose a sufficiently large $\omega^* > 0$ so that $\widehat{u}(\cdot, \omega)$ and $\widehat{f}(\cdot, \omega)$ are negligible for $|\omega| > \omega^*$. Also recall that the computation of $\widehat{u}(\cdot, \omega)$ for $\omega < 0$ is not necessary. Let M be a positive integer and define the discretization parameter $\Delta\omega$ of the frequency domain by the formula $\Delta\omega = \omega^*/M$, and introduce the mesh points $\omega_{j-\frac{1}{2}} = (j - \frac{1}{2})\Delta\omega, j = 1, \cdots, M$ on the interval $(0,\omega^*)$. Due to (2.2), the time-domain solution u of (1.1) will then be approximated by

$$u^h_{\omega^*,\Delta\omega}(x,t) = \frac{1}{\pi}\sum_{j=1}^{M} \widehat{u}_h(x,\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega.$$

We now try to estimate the convergence of $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ to $u(\cdot,t)$ for a fixed time t. Setting

$$u_{\omega^*}(x,t) = \frac{1}{\pi}\int_0^{\omega^*} \widehat{u}(x,\omega)e^{i\omega t}d\omega$$

and

$$u_{\omega^*,\Delta\omega}(x,t) = \frac{1}{\pi}\sum_{j=1}^{M} \widehat{u}(x,\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega,$$

we write

$$\begin{aligned} u(x,t) - u^h_{\omega^*,\Delta\omega}(x,t) &= (u(x,t) - u_{\omega^*}(x,t)) \\ &\quad + (u_{\omega^*}(x,t) - u_{\omega^*,\Delta\omega}(x,t)) \\ &\quad + \left(u_{\omega^*,\Delta\omega}(x,t) - u^h_{\omega^*,\Delta\omega}(x,t)\right) \\ &\equiv E_1(x,t) + E_2(x,t) + E_3(x,t). \end{aligned}$$

First, by Lemma 2.1 we get

$$\begin{aligned} \|E_1(\cdot,t)\| &\leq \tfrac{C}{\pi}\textstyle\int_{\omega>\omega^*} \|\widehat{u}(\cdot,\omega)\|d\omega \\ &\leq C\textstyle\int_{\omega>\omega^*} \frac{1}{\omega}\|\widehat{f}(\cdot,\omega)\|d\omega \\ &\leq C\textstyle\int_{\omega>\omega^*} \|\widehat{f}(\cdot,\omega)\|d\omega. \end{aligned} \tag{3.10}$$

Thus $\|E_1(\cdot,t)\| \to 0$ as $\omega^* \to \infty$.

We also have the following estimate for $\|E_2(\cdot,t)\|$, using the midpoint rule for the Fourier inversion formula.

$$
\begin{aligned}
\|E_2(\cdot,t)\|^2 &\leq \frac{C}{\pi^2}\int_\Omega \left|\int_0^{\omega^*}\widehat{u}(x,t)e^{i\omega t}d\omega - \sum_{j=1}^M \widehat{u}(x,\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega\right|^2 dx \\
&\leq C(\Delta\omega)^4\int_\Omega \left\|\frac{\partial^2\widehat{u}(x,\omega)e^{i\omega t}}{\partial\omega^2}\right\|^2_{L^2(0,\omega^*)} dx \\
&= C(\Delta\omega)^4\int_\Omega \left\|-\widehat{t^2u}(x,\cdot)+2t\widehat{tu}(x,\cdot)-t^2\widehat{u}(x,\cdot)\right\|^2_{L^2(0,\omega^*)} dx \\
&\leq C(\Delta\omega)^4\int_\Omega \Big\{\|\widehat{t^2u}(x,\cdot)\|^2_{L^2(0,\infty)} \\
&\quad + t^2\|\widehat{tu}(x,\cdot)\|^2_{L^2(0,\infty)} + t^4\|\widehat{u}(x,\cdot)\|^2_{L^2(0,\infty)}\Big\} dx \qquad (3.11) \\
&= C(\Delta\omega)^4\Big\{\|t^2u\|^2_{L^2((0,\infty);L^2(\Omega))} \\
&\quad + t^2\|tu\|^2_{L^2((0,\infty);L^2(\Omega))} + t^4\|u\|^2_{L^2((0,\infty);L^2(\Omega))}\Big\},
\end{aligned}
$$

where the last equality is due to the Parseval identity. Thus, if the assumption of Lemma 3.1 holds, then $\|E_2(\cdot,t)\| \to 0$ as $\Delta\omega \to 0$.

Finally, from Theorem 3.1, we have

$$
\begin{aligned}
\|E_3(\cdot,t)\| &\leq C\left\|\frac{1}{\pi}\sum_{j=1}^M\left(\widehat{u}_h(\cdot,\omega_{j-1/2})-\widehat{u}(\cdot,\omega_{j-1/2})\right)e^{it\omega_{j-1/2}}\Delta\omega\right\| \\
&\leq C\frac{\Delta\omega}{\pi}\sum_{j=1}^M\left\|\widehat{u}_h(\cdot,\omega_{j-1/2})-\widehat{u}(\cdot,\omega_{j-1/2})\right\| \qquad (3.12) \\
&\leq C\frac{\Delta\omega}{\pi}\sum_{j=1}^M h^2(\omega_{j-1/2}^2+\omega_{j-1/2}^{-2})\|\widehat{f}(\cdot,\omega_{j-1/2})\| \\
&\leq Ch^2\left\|(\omega^2+\omega^{-2})\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))}.
\end{aligned}
$$

Thus, if we assume that

$$
\left\|(\omega^2+\omega^{-2})\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))} \equiv \left[\int_0^\infty \left\|(\omega^2+\omega^{-2})\widehat{f}(\cdot,\omega)\right\|^2 d\omega\right]^{1/2} < \infty,
$$

then $\|E_3(\cdot,t)\| \to 0$ as $h \to 0$.

Combining the estimates (3.10), (3.11) and (3.12), and using Lemma 3.1 we have the *full error estimate.*

THEOREM 3.2. *Assume that the assumption of* Lemma 3.1 *holds for $k = 0, 1, 2$ and that*

$$
\left\|\left(\omega^2+\omega^{-2}\right)\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))} < \infty.
$$

Then $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ converges to $u(\cdot,t)$ for a fixed time $t>0$; moreover,

$$\begin{aligned}\|u(\cdot,t)-u^h_{\omega^*,\Delta\omega}(\cdot,t)\| \le C_1 \int_{\omega>\omega^*} &\|\widehat{f}(\cdot,\omega)\|\,d\omega \\ &+C_2(\Delta\omega)^2\sum_{k=0}^{2} t^{2-k}\sum_{l=0}^{k}\|\mathcal{P}(t)\|_{L^2(0,\infty)} \\ &+C_3\,h^2\left\|\left(\omega^2+\omega^{-2}\right)\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))},\end{aligned}$$

with $C_j, j=1,2,3$, dependent only on the domain Ω and the coefficients κ and λ.

Finally estimating $\|E_j(\cdot,\omega)\|_1$, for $j=1,2,3$, similarly as the above and using Lemma 3.2, we have the following error estimate.

THEOREM 3.3. *Assume that the assumption of* Lemma 3.2 *holds for $k=0,1,2$ and that*

$$\left\|\left(\omega^{-3/2}+\omega\right)\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))}<\infty.$$

Then $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ converges to $u(\cdot,t)$ for a fixed time $t>0$; moreover

$$\begin{aligned}\|u(\cdot,t)-u^h_{\omega^*,\Delta\omega}(\cdot,t)\|_1 \le C_1 \int_{\omega>\omega^*} &\|\widehat{f}(\cdot,\omega)\|\,d\omega \\ &+C_2(\Delta\omega)^2\sum_{k=0}^{2} t^{2-k}\left\{\left[\int_0^\infty t^{2k}\int_0^t \|f(\cdot,r)\|^2 dr\,dt\right]^{1/2}\right. \\ &\qquad\qquad\left.+\sum_{l=0}^{k}\|\mathcal{P}^l_k(t)\|_{L^2(0,\infty)}\right\} \\ &+C_3\,h\left\|\left(\omega^{-3/2}+\omega\right)\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))},\end{aligned}$$

with $C_j, j=1,2,3$, dependent only on the domain Ω and the coefficients κ and λ.

REMARK 3.2. We are interested in finding the solution $u(x,t)$ of Problem (1.1) for all $(x,t)\in\Omega\times J$, $J=[0,T]$. It is at our disposal how to extend $f(x,t)$ for $t>T$ in order to transform the space-time domain problem (1.1) into the frequency–space domain problem (2.3). For example, we may extend $f(x,t)$ by zero for $t>T$. But in this case, due to the

Neumann boundary condition, the solution may not decay as time grows. So we cannot expect that both of the second term of the right-hand sides of Theorem 3.2 and Theorem 3.3 are bounded for Problem (1.1) with this extended source. Instead, we alternatively may extend the given time-limited source function $f(x,t)$ for $t > T$ so that $f(x,t) = 0, \quad t > T_e$ for some $T_e > T$ and that

$$\widehat{f}(x,0) = \int_0^\infty f(x,t)dt = \int_0^{T_e} f(x,t)dt = 0, \quad x \in \Omega.$$

Then the solution $u(x,t)$ of Problem (1.1) with the extended source vanishes for all $t \geq T_c$ for some $T_c \geq T_e$ as the total source applied is zero. With this modification of Problem (1.1), each integral with respect to t in the last term of (3.11) is the integral over a finite interval. Therefore the integrals of both of the second term of the right-hand sides of Theorem 3.2 and Theorem 3.3 are over a finite interval, so they are bounded in view of Remark 3.1. Finally, note that the solution $u(x,t_0), \ t_0 \leq T$ of Problem (1.1) depends only on $u(x,t)$ and $f(x,t), \quad t < t_0$. So the solution $u(x,t)$ of the modified problem, in fact, is the same as the solution of the original problem for all $(x,t) \in \Omega \times J$.

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