

## E-DEPTHES AND T-CODEPHTHS OF MODULES

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**ABSTRACT.** We investigate relationships of E-depths and T-codepths of modules in a short exact sequence. We give E-depths and T-codepths of some modules.

### 0. Introduction

Rees [Re] introduced the notion of depth and E-depth and Matlis [Mt] defined the concept of codepth. The depth and codepth of modules play an important roll in commutative ring theory. The depth of Noetherian modules give a criterion of Cohen-Macaulay modules and the codepth of Artinian modules also characterize Co-Cohen-Macaulay modules[TZ]. Ooishi [O] studied the relationship of codepths of modules in a short exact sequence. It is well-known the basic relationship of depths of modules in a short exact sequence [BH, 1.2.9].

Strooker[St] introduced the concept of E-depth and T-codepth which are the extended notion of depth and codepth and gave some relationships between E-depth and T-codepth of a module (see *Remark 1.2*).

The purpose of this paper is to investigate some properties of E-depth and T-codepth of modules. In section 1, we give relationships of E-depths and T-codepths of modules in a short exact sequence. In section 2, we study E-depths and T-codepths of certain modules.

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## 1. Relationships of E-depths and T-codepths

Throughout this note,  $R$  is a commutative Noetherian ring with non-zero identity, and  $M$  an  $R$ -module and  $I$  an ideal of  $R$ .

**DEFINITION 1.1.** [St] The least integer  $i$  for which  $\text{Ext}_R^i(R/I, M) \neq 0$  is called the *E-depth* of  $M$  in  $I$ , denoted by  $E\text{-dp}(I, M)$ , and the least integer  $j$  for which  $\text{Tor}_j^R(R/I, M) \neq 0$  the *T-codepth* of  $M$  in  $I$ , denoted by  $T\text{-codp}(I, M)$ .

Thus E-depth and T-codepth are nonnegative integers or, if such  $i$ 's do not exist, they are  $\infty$ . For the null module  $M$  and every ideal  $I$  we denote  $E\text{-dp}(I, M) = T\text{-codp}(I, M) = \infty$ .

**REMARK 1.2.** (1) [St, p. 91] In general  $E\text{-dp}(I, M) \geq \text{depth}(I, M)$  for an  $R$ -module  $M$ .

(2) [St, 5.3.9] If  $M$  is a finite  $R$ -module such that  $M/IM \neq 0$ , then  $E\text{-dp}(I, M)$  is the length of a maximal  $M$ -sequence in  $I$ . That is  $E\text{-dp}(I, M) = \text{depth}(I, M)$ .

(3) [O, 3.11] If  $M$  is an Artinian  $R$ -module such that  $(0 :_M I) \neq 0$ , then  $T\text{-codp}(I, M)$  is the length of a maximal  $M$ -cosequence in  $I$ . That is  $T\text{-codp}(I, M) = \text{codepth}(I, M)$ .

(4) [St, 6.1.10] If  $(R, \mathfrak{m})$  is a  $d$ -dimensional Noetherian local ring. Then  $E\text{-dp}(\mathfrak{m}, M)$  is finite if and only if  $T\text{-codp}(\mathfrak{m}, M)$  is finite. In this case  $E\text{-dp}(\mathfrak{m}, M) + T\text{-codp}(\mathfrak{m}, M) \leq d$ .

**LEMMA 1.3.** Let  $L, M, N$  be  $R$ -modules and  $I$  an ideal of  $R$ . Consider the following short exact sequence.

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then we have the following.

- (1)  $T\text{-codp}(I, M) \geq \min\{T\text{-codp}(I, L), T\text{-codp}(I, N)\}$ .
- (2)  $T\text{-codp}(I, L) \geq \min\{T\text{-codp}(I, M), T\text{-codp}(I, N) - 1\}$ .
- (3)  $T\text{-codp}(I, N) \geq \min\{T\text{-codp}(I, M), T\text{-codp}(I, L) + 1\}$ .

*Proof.* (1) From the given exact sequence we have a long exact sequence

$$\dots \longrightarrow \text{Tor}_{i+1}^R(R/I, N) \longrightarrow \text{Tor}_i^R(R/I, L) \longrightarrow \text{Tor}_i^R(R/I, M) \\ \longrightarrow \text{Tor}_i^R(R/I, N) \longrightarrow \text{Tor}_{i-1}^R(R/I, L) \longrightarrow \dots$$

Hence we obtain that  $\text{Tor}_i^R(R/I, M) = 0$  if  $\text{Tor}_i^R(R/I, L) = \text{Tor}_i^R(R/I, N) = 0$ . Therefore the first inequality follows from the definition of T-codepth.

(2) (3) Completely analogous arguments show that the second and third inequalities hold.  $\square$

The next Proposition 1.4 is a slightly extended version of Ooishi. He proved it under the condition of Artinian modules.

**PROPOSITION 1.4.** [cf. O, 3.16] *With the same hypotheses as in 1.3, we have the following.*

- (1) *If  $T\text{-codp}(I, M) < T\text{-codp}(I, L)$  then  $T\text{-codp}(I, M) = T\text{-codp}(I, N)$ .*
- (2) *If  $T\text{-codp}(I, M) > T\text{-codp}(I, L)$  then  $T\text{-codp}(I, N) = T\text{-codp}(I, L) + 1$ .*
- (3) *If  $T\text{-codp}(I, M) = T\text{-codp}(I, L)$  then  $T\text{-codp}(I, M) \leq T\text{-codp}(I, N)$ .*
- (4) *If  $T\text{-codp}(I, M) < T\text{-codp}(I, N)$  then  $T\text{-codp}(I, M) = T\text{-codp}(I, L)$ .*
- (5) *If  $T\text{-codp}(I, M) > T\text{-codp}(I, N)$  then  $T\text{-codp}(I, N) = T\text{-codp}(I, L) + 1$ .*
- (6) *If  $T\text{-codp}(I, M) = T\text{-codp}(I, N)$  then  $T\text{-codp}(I, M) \leq T\text{-codp}(I, L) + 1$ .*

*Proof.* (1) From the hypothesis and 1.3(1)(3) we have

$$T\text{-codp}(I, L) > T\text{-codp}(I, M) \geq T\text{-codp}(I, N) \geq T\text{-codp}(I, M).$$

(2) From the hypothesis and 1.3(2)(3) we have

$$\begin{aligned} T\text{-codp}(I, M) &> T\text{-codp}(I, L) \geq T\text{-codp}(I, N) - 1 \\ &\geq \min\{T\text{-codp}(I, L), T\text{-codp}(I, M) - 1\} \geq T\text{-codp}(I, L). \end{aligned}$$

(3) By 1.3(3) it is clear.

(4) From the hypothesis and 1.3(1)(2) we have

$$T\text{-codp}(I, N) > T\text{-codp}(I, M) \geq T\text{-codp}(I, L) \geq T\text{-codp}(I, M).$$

(5) From the hypothesis and 1.3(2)(3) we have

$$\begin{aligned} T\text{-codp}(I, M) &> T\text{-codp}(I, N) \geq T\text{-codp}(I, L) + 1 \\ &\geq \min\{T\text{-codp}(I, M) + 1, T\text{-codp}(I, N)\} \geq T\text{-codp}(I, N). \end{aligned}$$

(6) From the hypothesis and 1.3(2) we have

$$\begin{aligned} T\text{-codp}(I, L) + 1 &\geq \min\{T\text{-codp}(I, M) + 1, T\text{-codp}(I, N)\} \\ &\geq T\text{-codp}(I, N) = T\text{-codp}(I, M). \end{aligned}$$

□

**THEOREM 1.5.** *With the same hypotheses as in 1.3, there are only the following cases.*

- (i)  $T\text{-codp}(I, L) = T\text{-codp}(I, M) \leq T\text{-codp}(I, N)$ .
- (ii)  $T\text{-codp}(I, L) \geq T\text{-codp}(I, M) = T\text{-codp}(I, N)$ .
- (iii)  $T\text{-codp}(I, M) \geq T\text{-codp}(I, N) = T\text{-codp}(I, L) + 1$ .

*Proof.* For the T-codepths of modules in the short exact sequence, there may be only the following case involving strict inequalities among three modules :  $T\text{-codp}(I, L) = T\text{-codp}(I, N) - 1 < T\text{-codp}(I, N) < T\text{-codp}(I, M)$  and the other cases do not hold. In fact, we show the following cases.

- (1) By 1.4(2),  $T\text{-codp}(I, L) < T\text{-codp}(I, M) < T\text{-codp}(I, N)$  does not hold.
- (2) By 1.4(1),  $T\text{-codp}(I, M) < T\text{-codp}(I, L) < T\text{-codp}(I, N)$  does not hold.
- (3) By 1.4(5),  $T\text{-codp}(I, N) < T\text{-codp}(I, L) < T\text{-codp}(I, M)$  does not hold.
- (4) By 1.4(1),  $T\text{-codp}(I, M) < T\text{-codp}(I, N) < T\text{-codp}(I, L)$  does not hold.
- (5) By 1.4(1),  $T\text{-codp}(I, N) < T\text{-codp}(I, M) < T\text{-codp}(I, L)$  does not hold.

In particular, if  $T\text{-codp}(I, L) < T\text{-codp}(I, N) < T\text{-codp}(I, M)$ , then we get

$$T\text{-codp}(I, L) + 1 = T\text{-codp}(I, N) < T\text{-codp}(I, M)$$

by 1.4(5).

On the other hand,

- (6) By 1.4(2),  $T\text{-codp}(I, L) = T\text{-codp}(I, N) < T\text{-codp}(I, M)$  does not hold.
- (7) By 1.4(1),  $T\text{-codp}(I, M) < T\text{-codp}(I, L) = T\text{-codp}(I, N)$  does not hold.
- (8) By 1.4(5),  $T\text{-codp}(I, N) < T\text{-codp}(I, L) = T\text{-codp}(I, M)$  does not hold.

If  $T\text{-codp}(I, L) < T\text{-codp}(I, M)$  or  $T\text{-codp}(I, N) < T\text{-codp}(I, M)$ , then we have

$$T\text{-codp}(I, L) + 1 = T\text{-codp}(I, N)$$

by 1.4(2)(5).

From the above arguments, we have the conclusions.  $\square$

The following Lemma 1.6 was given by Bruns and Herzog under the condition of finite modules.

LEMMA 1.6. [cf. BH, 1.2.9] *With the same hypotheses as in 1.3, we have the following.*

- (1)  $E\text{-dp}(I, M) \geq \min\{E\text{-dp}(I, L), E\text{-dp}(I, N)\}.$
- (2)  $E\text{-dp}(I, L) \geq \min\{E\text{-dp}(I, M), E\text{-dp}(I, N) + 1\}.$
- (3)  $E\text{-dp}(I, N) \geq \min\{E\text{-dp}(I, L) - 1, E\text{-dp}(I, M)\}.$

*Proof.* (1) The given exact sequence induces a long exact sequence

$$\cdots \longrightarrow \text{Ext}_R^{i-1}(R/I, N) \longrightarrow \text{Ext}_R^i(R/I, L) \longrightarrow \text{Ext}_R^i(R/I, M) \\ \longrightarrow \text{Ext}_R^i(R/I, N) \longrightarrow \text{Ext}_R^{i+1}(R/I, L) \longrightarrow \cdots$$

The similar arguments as in 1.3 show that the conclusions hold.  $\square$

PROPOSITION 1.7. *With the same hypotheses as in 1.3, we have the following.*

- (1) If  $E\text{-dp}(I, M) < E\text{-dp}(I, L)$  then  $E\text{-dp}(I, M) = E\text{-dp}(I, N).$
- (2) If  $E\text{-dp}(I, M) > E\text{-dp}(I, L)$  then  $E\text{-dp}(I, N) = E\text{-dp}(I, L) - 1.$
- (3) If  $E\text{-dp}(I, M) = E\text{-dp}(I, L)$  then  $E\text{-dp}(I, M) \leq E\text{-dp}(I, N) + 1.$
- (4) If  $E\text{-dp}(I, M) < E\text{-dp}(I, N)$  then  $E\text{-dp}(I, M) = E\text{-dp}(I, L).$
- (5) If  $E\text{-dp}(I, M) > E\text{-dp}(I, N)$  then  $E\text{-dp}(I, N) = E\text{-dp}(I, L) - 1.$
- (6) If  $E\text{-dp}(I, M) = E\text{-dp}(I, N)$  then  $E\text{-dp}(I, M) \leq E\text{-dp}(I, L).$

*Proof.* (1) From the hypothesis and 1.6(1)(3) we have

$$E\text{-dp}(I, L) > E\text{-dp}(I, M) \geq E\text{-dp}(I, N) \geq E\text{-dp}(I, M).$$

(2) From the hypothesis and 1.6(2)(3) we have

$$E\text{-dp}(I, M) > E\text{-dp}(I, L) \geq E\text{-dp}(I, N) + 1 \geq E\text{-dp}(I, L).$$

(3) By 1.6(3) it is clear.

(4) From the hypothesis and 1.6(1)(2) we have

$$E\text{-dp}(I, N) > E\text{-dp}(I, M) \geq E\text{-dp}(I, L) \geq E\text{-dp}(I, M).$$

(5) From the hypothesis and 1.6(3)(2) we have

$$E\text{-dp}(I, M) > E\text{-dp}(I, N) \geq E\text{-dp}(I, L) - 1 \geq E\text{-dp}(I, N).$$

(6) From the hypothesis and 1.6(2) we have

$$E\text{-dp}(I, L) \geq E\text{-dp}(I, M). \quad \square$$

**THEOREM 1.8.** *With the same hypotheses as in 1.3, there are only the following cases.*

- (i)  $E\text{-dp}(I, L) = E\text{-dp}(I, M) \leq E\text{-dp}(I, N)$ .
- (ii)  $E\text{-dp}(I, L) \geq E\text{-dp}(I, M) = E\text{-dp}(I, N)$ .
- (iii)  $E\text{-dp}(I, M) \geq E\text{-dp}(I, L) = E\text{-dp}(I, N) + 1$ .

*Proof.* For E-depths of modules of the short exact sequence, there may be only the following case involving strict inequalities among three modules :  $E\text{-dp}(I, N) = E\text{-dp}(I, L) - 1 < E\text{-dp}(I, L) < E\text{-dp}(I, M)$  and the other cases do not hold. In fact, we prove the following.

- (1) By 1.7(2),  $E\text{-dp}(I, L) < E\text{-dp}(I, M) < E\text{-dp}(I, N)$  does not hold.
- (2) By 1.7(1),  $E\text{-dp}(I, M) < E\text{-dp}(I, L) < E\text{-dp}(I, N)$  does not hold.
- (3) By 1.7(5),  $E\text{-dp}(I, L) < E\text{-dp}(I, N) < E\text{-dp}(I, M)$  does not hold.
- (4) By 1.7(1),  $E\text{-dp}(I, M) < E\text{-dp}(I, N) < E\text{-dp}(I, L)$  does not hold.
- (5) By 1.7(1),  $E\text{-dp}(I, N) < E\text{-dp}(I, M) < E\text{-dp}(I, L)$  does not hold.

In particular, if  $E\text{-dp}(I, N) < E\text{-dp}(I, L) < E\text{-dp}(I, M)$ , then we get

$$E\text{-dp}(I, N) + 1 = E\text{-dp}(I, L) < E\text{-dp}(I, M)$$

by 1.7(2).

On the other hand,

- (6) By 1.7(2),  $E\text{-dp}(I, L) = E\text{-dp}(I, N) < E\text{-dp}(I, M)$  does not hold.
- (7) By 1.7(1),  $E\text{-dp}(I, M) < E\text{-dp}(I, L) = E\text{-dp}(I, N)$  does not hold.
- (8) By 1.7(2),  $E\text{-dp}(I, L) < E\text{-dp}(I, M) = E\text{-dp}(I, N)$  does not hold.

If  $E\text{-dp}(I, L) < E\text{-dp}(I, M)$  or  $E\text{-dp}(I, M) > E\text{-dp}(I, N)$ , then we have

$$E\text{-dp}(I, L) = E\text{-dp}(I, N) + 1$$

by 1.7(2)

From the above arguments, we have the conclusions. □

## 2. E-depths and T-codepths of modules

In this section, we calculate E-depths and T-codepths of certain modules.

**LEMMA 2.1.** [Sp, 4.4 and Rt, 8.10] *Let  $M$ ,  $M_j$  and  $N$  be  $R$ -modules. Assume that  $M \cong \bigoplus_j M_j$  for some directed system  $J$ . Then we have the following.*

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- (1)  $\mathrm{Ext}_R^i(N, \bigoplus_j M_j) \cong \bigoplus_j \mathrm{Ext}_R^i(N, M_j)$ , if  $N$  is finite.
- (2)  $\mathrm{Tor}_i^R(N, \bigoplus_j M_j) \cong \bigoplus_j \mathrm{Tor}_i^R(N, M_j)$ .

The next Theorem 2.2 is similar to the result given by Sharp [cf. Sp, 4.5] in the Cousin complex. We extend his result.

**THEOREM 2.2.** *Let  $M$  and  $N$  be  $R$ -modules. Assume  $M \cong \bigoplus_{\mathfrak{p} \in S} M_{\mathfrak{p}}$ , where  $S$  is a subset of prime ideals of  $R$ . Suppose that  $(0 :_R N) \not\subset \mathfrak{p}$  for all  $\mathfrak{p} \in S$ . Then we have*

- (1)  $\mathrm{Ext}_R^i(N, M) = 0$  for all  $i$ , if  $N$  is finite.
- (2)  $\mathrm{Tor}_i^R(N, M) = 0$  for all  $i$ .

*Proof.* (1) By 2.1 and [Rt, 11.65], we have

$$\begin{aligned} \mathrm{Ext}_R^i(N, M) &\cong \mathrm{Ext}_R^i(N, \bigoplus_{\mathfrak{p} \in S} M_{\mathfrak{p}}) \cong \bigoplus_{\mathfrak{p} \in S} \mathrm{Ext}_R^i(N, M_{\mathfrak{p}}) \\ &\cong \bigoplus_{\mathfrak{p} \in S} \mathrm{Ext}_{R_{\mathfrak{p}}}^i(N_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0, \end{aligned}$$

since  $N_{\mathfrak{p}} = 0$  for all  $\mathfrak{p} \in S$ .

(2) By 2.1 and [Rt, 11.63], we have

$$\begin{aligned} \mathrm{Tor}_i^R(N, M) &\cong \mathrm{Tor}_i^R(N, \bigoplus_{\mathfrak{p} \in S} M_{\mathfrak{p}}) \cong \bigoplus_{\mathfrak{p} \in S} \mathrm{Tor}_i^R(N, M_{\mathfrak{p}}) \\ &\cong \bigoplus_{\mathfrak{p} \in S} \mathrm{Tor}_{i_{\mathfrak{p}}}^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0, \end{aligned}$$

since  $N_{\mathfrak{p}} = 0$  for all  $\mathfrak{p} \in S$ .  $\square$

Recall that a module  $P$  is a *generator* (resp.  $E$  is a *cogenerator*) if for every  $R$ -module  $N$  there is a set  $A$  and  $R$ -epimorphism

$$P^A \longrightarrow N \longrightarrow 0,$$

respectively, if each  $R$ -module  $N$  can be embedded in a product of copies of  $E$

$$0 \longrightarrow N \longrightarrow E^A.$$

**THEOREM 2.3.** *Let  $M$  and  $M'$  be  $R$ -modules,  $P$  a finitely generated projective  $R$ -module,  $E$  an injective  $R$ -module and  $I$  an ideal of  $R$ . Suppose that  $M \cong \bigoplus_{\mathfrak{q} \in S} M_{\mathfrak{q}}$ , where  $S$  is a subset of prime ideals of  $R$ . Then we have*

$$\begin{aligned}
 (1) \quad & \left\{ \begin{array}{ll} E\text{-}dp(I, M) = \infty, & \text{if } I \not\subset q \text{ for all } q \in S. \\ E\text{-}dp(I, M') = 0, & \text{if } I \subset q \text{ for some } q \in \text{Ass}(M'). \end{array} \right. \\
 (2) \quad & \left\{ \begin{array}{ll} T\text{-}codp(I, M) = \infty, & \text{if } I \not\subset q \text{ for all } q \in S. \\ T\text{-}codp(I, M) = 0, & \text{if } I \subset q \text{ for some } q \in S \text{ and} \\ & M \text{ is finite.} \\ T\text{-}codp(I, M') = 0, & \text{if } I \subset \text{Jac}(R) \text{ and } M' \text{ is finite.} \\ E\text{-}dp(I, \text{Hom}(P, M)) = \infty, & \text{if } I \not\subset q \text{ for all } q \in S. \\ E\text{-}dp(I, \text{Hom}(P, M')) = 0, & \text{if } I \subset q \text{ for some } q \in \text{Ass}(M') \\ & \text{and } P \text{ is a generator.} \\ E\text{-}dp(I, \text{Hom}(M, E)) = \infty, & \text{if } I \not\subset q \text{ for all } q \in S. \\ E\text{-}dp(I, \text{Hom}(M, E)) = 0, & \text{if } I \subset q \text{ for some } q \in S, \\ & \text{and } M \text{ is finite} \\ & \text{and } E \text{ is a cogenerator.} \\ E\text{-}dp(I, \text{Hom}(M', E)) = 0, & \text{if } I \subset \text{Jac}(R), \text{ and } M' \text{ is finite} \\ & \text{and } E \text{ is a cogenerator.} \\ T\text{-}codp(I, \text{Hom}(P, M)) = \infty, & \text{if } I \not\subset q \text{ for all } q \in S. \\ T\text{-}codp(I, \text{Hom}(P, M)) = 0, & \text{if } I \subset q \text{ for some } q \in S, \\ & \text{and } M \text{ is finite} \\ & \text{and } P \text{ is a generator.} \\ T\text{-}codp(I, \text{Hom}(P, M')) = 0, & \text{if } I \subset \text{Jac}(R), \\ & \text{and } M' \text{ is finite} \\ & \text{and } P \text{ is a generator.} \\ T\text{-}codp(I, \text{Hom}(M, E)) = \infty, & \text{if } I \not\subset q \text{ for all } q \in S. \\ T\text{-}codp(I, \text{Hom}(M', E)) = 0, & \text{if } I \subset q \text{ for some } q \in \text{Ass}(M') \\ & \text{and } E \text{ is a cogenerator.} \end{array} \right. \\
 (4) \quad & \left\{ \begin{array}{ll} \end{array} \right.
 \end{aligned}$$

*Proof.* (1) If  $I \not\subset q$  for all  $q \in S$ , This is an easy conclusion of 2.2(1).  
 Next, if  $I \subset q$  for some  $q \in \text{Ass}_R(M')$ , then we get

$$\text{Hom}_R(R/I, M') \cong \text{Ann}_{M'}(I) \supset \text{Ann}_{M'}(q) \neq 0.$$

(2) When  $I \not\subset q$  for all  $q \in S$ , from 2.2(2) we have the conclusion.  
 If  $I \subset q$  for some  $q \in S$  and  $M$  is finite, then we have

$$R/I \otimes_R M \cong \bigoplus_{q' \in S} R/I \otimes_R M_{q'} \cong \bigoplus_{q' \in S} (R/I)_{R_{q'}} \otimes_{R_{q'}} M_{q'} \supset M_q/(I_q M_q) \neq 0,$$

by [Rt, 11.63] and Nakayama lemma.

For the finite module  $M'$ , it easily follows from Nakayama lemma.

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(3) If  $I \not\subset q$  for all  $q \in S$ , then by 2.1 and [Rt, 9.21 and 11.65] we have for all  $i$

$$\begin{aligned} \mathrm{Ext}_R^i(R/I, \mathrm{Hom}(P, M)) &\cong \mathrm{Ext}_R^i(R/I, \mathrm{Hom}(P, \bigoplus_{q \in S} M_q)) \\ &\cong \bigoplus_{q \in S} \mathrm{Ext}_R^i(R/I, \mathrm{Hom}(P, M_q)) \cong \bigoplus_{q \in S} \mathrm{Ext}_R^i(R/I \otimes P, M_q) \\ &\cong \bigoplus_{q \in S} \mathrm{Ext}_R^i(P/IP, M_q) \cong \bigoplus_{q \in S} \mathrm{Ext}_{R_q}^i(P_q/I_q P_q, M_q) = 0, \end{aligned}$$

since  $P_q/I_q P_q = 0$ .

If  $I \subset q$  for some  $q \in \mathrm{Ass}(M')$ , then by [AF, 17.5] we have

$$\begin{aligned} \mathrm{Hom}(R/I, \mathrm{Hom}(P, M')) &\cong \mathrm{Hom}(R/I \otimes P, M') \\ &\cong \mathrm{Hom}(P, \mathrm{Hom}(R/I, M')) \neq 0, \end{aligned}$$

since  $\mathrm{Hom}(R/I, M') \cong \mathrm{Ann}_{M'} I \supset \mathrm{Ann}_{M'} q \neq 0$  and  $P$  is a generator.

For the third assertion, when  $I \not\subset q$  for all  $q \in S$ , by [CE, p.120, 5.1] or [Rt, 11.54] we have

$$\mathrm{Ext}_R^i(R/I, \mathrm{Hom}(M, E)) \cong \mathrm{Hom}(\mathrm{Tor}_i^R(R/I, M), E) = 0,$$

since  $\mathrm{Tor}_i^R(R/I, M) = 0$  by (2).

Next, if  $M, M'$  are finite and  $E$  is a cogenerator, and if  $I \subset q$  for some  $q \in S$  or  $I \subset \mathrm{Jac}(R)$ , then the fourth and the fifth assertions follow from (2) and [AF, 18.14], since

$$\mathrm{Hom}(R/I, \mathrm{Hom}(N, E)) \cong \mathrm{Hom}(N/IN, E)$$

for every  $R$ -module  $N$ .

(4) If  $I \not\subset q$  for all  $q \in S$ , then we have by 2.1 and [Rt, 11.65 and 11.63]

$$\begin{aligned} \mathrm{Tor}_i^R(R/I, \mathrm{Hom}(P, M)) &\cong \bigoplus_{q \in S} \mathrm{Tor}_i^R(R/I, \mathrm{Hom}(P, M_q)) \\ &\cong \bigoplus_{q \in S} \mathrm{Tor}_i^{R_q}(R_q/I_q, \mathrm{Hom}_{R_q}(P_q, M_q)) = 0 \text{ for all } i, \end{aligned}$$

since  $P$  is finite and  $R_q/I_q = 0$ .

When  $M, M'$  are finite and  $P$  is a generator, if  $I \subset q$  for some  $q \in S$  or  $I \subset \mathrm{Jac}(R)$ , then the second and the third assertions follow from (2)

and [AF, 17.5], since from [I, 1.1]

$$R/I \otimes \text{Hom}(P, N) \cong \text{Hom}(P, R/I \otimes N) \cong \text{Hom}(P, N/IN)$$

for every  $R$ -module  $N$ .

For the fourth part, if  $I \not\subset q$  for all  $q \in S$ , then we have by [Rt, 9.51]

$$\text{Tor}_i^R(R/I, \text{Hom}(M, E)) \cong \text{Hom}(\text{Ext}_R^i(R/I, M), E) = 0 \text{ for all } i,$$

since  $\text{Ext}_R^i(R/I, M) = 0$  by (1).

On the other hand, if there is  $q \in \text{Ass}(M')$  such that  $I \subset q$  and  $E$  is a cogenerator, then by [Rt, 9.51] and [AF, 18.14] we have

$$R/I \otimes \text{Hom}(M', E) \cong \text{Hom}(\text{Hom}(R/I, M'), E) \neq 0,$$

since  $\text{Hom}(R/I, M') \neq 0$ . □

In Corollary 2.4, we study depths and codepths of modules of generalized fractions. For an finitely generated  $R$ -module  $M$ , consider the following triangular subsets(cf. [C]).

$$(U_h)_n = \{(a_1, \dots, a_n) \in R^n : ht_M(a_1, \dots, a_i)R \geq i \ (1 \leq i \leq n)\} \quad \text{and}$$

$$(U_s)_n = \{(a_1, \dots, a_n) \in R^n : \dim M/(a_1, \dots, a_i)M = d - i \ (1 \leq i \leq n)\}.$$

Then by [SZ] we can construct the following modules of generalized fractions

$$(U_h)_n^{-n}M = \left\{ \frac{m}{(a_1, \dots, a_n)} : (a_1, \dots, a_n) \in (U_h)_n \text{ and } m \in M \right\} \text{ and}$$

$$(U_s)_n^{-n}M = \left\{ \frac{m}{(a_1, \dots, a_n)} : (a_1, \dots, a_n) \in (U_s)_n \text{ and } m \in M \right\}.$$

**COROLLARY 2.4.** Let  $M$  be a finitely generated  $R$ -module with finite dimension  $d$ ,  $P$  a generator and  $E$  a cogenerator.

Then, for  $0 \leq n \leq d$ , we have the following.

- (1) 
$$\begin{aligned} E\text{-dp}(I, (U_h)_{n+1}^{-n-1}M) &= T\text{-codp}(I, (U_h)_{n+1}^{-n-1}M) \\ &= E\text{-dp}(I, \text{Hom}(P, (U_h)_{n+1}^{-n-1}M)) = T\text{-codp}(I, \text{Hom}(P, (U_h)_{n+1}^{-n-1}M)) \\ &= E\text{-dp}(I, \text{Hom}((U_h)_{n+1}^{-n-1}M, E)) = T\text{-codp}(I, \text{Hom}((U_h)_{n+1}^{-n-1}M, E)) \\ &= \begin{cases} \infty, & \text{if } I \not\subset q \text{ for all } q \in \text{Supp}(M) \text{ such that } ht_M q = n. \\ 0, & \text{if } I \subset q \text{ for some } q \in \text{Supp}(M) \text{ such that } ht_M q = n. \end{cases} \end{aligned}$$
- (2) In addition, suppose that  $M$  is satisfied  $d = ht_M q + \dim_R(M/qM)$  for all  $q \in \text{Supp}(M)$ . Put  $Q = \{q \in \text{Supp}(M) : d = ht_M q + \dim_R(M/qM)\}$ . Then we have

$$\begin{aligned}
 & E\text{-}dp(I, (U_s)_{n+1}^{-n-1}M) = T\text{-}dp(I, (U_s)_{n+1}^{-n-1}M) \\
 & = E\text{-}dp(I, \text{Hom}(P, (U_s)_{n+1}^{-n-1}M)) = T\text{-}dp(I, \text{Hom}(P, (U_s)_{n+1}^{-n-1}M)) \\
 & = E\text{-}dp(I, \text{Hom}((U_s)_{n+1}^{-n-1}M, E)) = T\text{-}dp(I, \text{Hom}((U_s)_{n+1}^{-n-1}M, E)) \\
 & = \begin{cases} \infty, & \text{if } I \not\subset q \text{ for all } q \in Q, \\ 0, & \text{if } I \subset q \text{ for some } q \in Q. \end{cases}
 \end{aligned}$$

*Proof.* From [C, 2.11, 2.12 and 3.3], we have

$$\begin{aligned}
 \text{Ass}((U_h)_{n+1}^{-n-1}M) &= \{q \in \text{Supp}(M) : ht_M q = n\}, \\
 \text{Ass}((U_s)_{n+1}^{-n-1}M) &= \{q \in \text{Supp}(M) : ht_M q = n \text{ and } \dim M/qM = d - n\} \\
 (U_h)_{n+1}^{-n-1}M &\cong \bigoplus_{ht_M q = n} ((U_h)_{n+1}^{-n-1}M)_q \text{ and} \\
 (U_s)_{n+1}^{-n-1}M &\cong \bigoplus_{\substack{ht_M q = n, \\ \dim M/qM = d - n}} ((U_s)_{n+1}^{-n-1}M)_q.
 \end{aligned}$$

Hence the conclusions easily follow from 2.3, since  $P$  is a generator and  $E$  is a cogenerator.  $\square$

**EXAMPLE 2.5.** In the above Corollary 2.4(1), there is an ideal  $I$  of  $R$  such that  $ht_M I < n$  but  $I \not\subset q$  for all  $q \in \text{Supp}(M)$  with  $ht_M q = n$ . Let  $R = k[X, Y, Z]/(X) \cap (Y, Z) = k[x, y, z]$  and  $I = (y, z)$ . Then  $R$  is a 2-dimensional Noetherian local ring and  $ht_R I = 0$ . Since  $R/(y, z) = k[X]$ , if  $I \subsetneq q$  for some  $q \in \text{Supp}(R)$  then  $q$  must be the maximal ideal of  $R$ .

**REMARK 2.6.** In the above Corollary 2.4, if  $n > d$  then all modules of generalized fractions  $U_{n+1}^{-n-1}M$ , where  $U_{n+1}$  is a triangular subset of  $R^{n+1}$ , are zero by [HS, 3.1]. Thus in this case their E-depth and T-codepth are  $\infty$ .

**REMARK 2.7.** Let  $(R, m)$  be a Noetherian local ring,  $M$  a finitely generated  $R$ -module with the finite dimension  $d$  and  $0 \leq n < d$ . Then the non-zero module of generalized fractions  $(U_h)_{n+1}^{-n-1}M$  is NOT both Artinian and Noetherian.

*Proof.* If  $(U_h)_{n+1}^{-n-1}M$  is finitely generated then

$$E\text{-}dp(m, (U_h)_{n+1}^{-n-1}M) = \text{depth}(m, (U_h)_{n+1}^{-n-1}M) = \infty$$

by 2.4 and [St, 5.3.9]. Hence  $(U_h)_{n+1}^{-n-1}M = 0$  by [Mm, p. 101 corollary].

If  $(U_h)_{n+1}^{-n-1}M$  is Artinian then  $\text{Ass}((U_h)_{n+1}^{-n-1}M) \subset \text{Max}(R)$ . This contradicts to [C, 2.11].  $\square$

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