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A Note on Weak Law of Large Numbers for $L^1(R)$ [1]

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Abstract

In this paper weak laws of large numbers are obtained for random variables in $L^1(R)$ which satisfy a compact uniform integrability condition.

Key Words and Phrases: weak law of large numbers, random variables in $L^1(R)$, compact uniform integrability.

1. Introduction

For identically distributed random elements $\{X_k\}$ in a separable normed linear space with $E\|X_1\| < \infty$, Taylor(1972) obtained $\left\|\frac{1}{n}\sum_{k=1}^{n} X_k - EX_1\right\| \to 0$ in probability if and only if $\left|\frac{1}{n}\sum_{k=1}^{n} f(X_k) - Ef(X_1)\right| \to 0$ in probability for each continuous linear functional f. Taylor and Wei(1979) obtained a more general result for a sequence of tight random elements. In particular, for tight random elements $\{X_k\}$ with $\sup_k E\|X_k\|^p < \infty$ $(p > 1)$, $\left\|\frac{1}{n}\sum_{k=1}^{n} X_k\right\| \to 0$ in probability if and only if $\left|\frac{1}{n}\sum_{k=1}^{n} f(X_k)\right| \to 0$ in probability for each continuous linear functional f. Wei and Taylor(1987) proved similar results for weighted sums, that is, $\|\sum_{k=1}^{n} a_{nk}X_k\| \to 0$ in probability if and only if $|\sum_{k=1}^{n} a_{nk}f(X_k)| \to 0$ in probability for each continuous linear functional f where $\{a_{nk}\}$ is a Toeplitz sequence. Wang and Rao(1987) obtained a weak law of large numbers by combining the tightness and moment conditions into a condition for compact uniformly integrable random elements.

This paper concentrates on weak law of large numbers of random elements in the space $L^1(R)$ with the L^1 norm. Some basic properties of random elements in $L^1(R)$ are introduced and weak laws of large numbers are obtained for compact uniformly integrable random elements in $L^1(R)$.

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2. Preliminaries

Let (Ω, A, μ) be a measure space. The space $L^1 = L^1(\Omega, A, \mu, R)$ is the set of all μ-equivalence classes of A measurable function $x : \Omega \to R$ such that $\int_\Omega |x| d\mu < \infty$. The norm for L^1 is defined by $\|x\| = \int_\Omega |x| d\mu$. For the space $L^1(R)$ of this paper, $\Omega = R$, A =the Borel subsets and μ will be the Lebesgue measure on R. Let (Ω, A, P) be a probability space and let X be a function from Ω into a Banach space E. If $X^{-1}(B) \in A$ for every Borel set $B \in B(E)$, then X is said to be a random element in E. The following characterization of random elements in $L^1(R)$ is illustrative and useful in obtaining later results.

Lemma 2.1 (a) Let X be a function from $R \times \Omega$ into R such that (i) for each, $t \in R$ $X(t, \cdot) : w \to X(t, w)$ is a random variable, (ii) for each $w \in \Omega, X(\cdot, w) : t \to X(t, w)$ is a Riemann integrable function. If for each $w \in \Omega, X(\cdot, w)$ is identified with $\tilde{X}(\cdot, w)$, the equivalence class of $X(\cdot, w)$, then $\tilde{X}$ is a random element in $L^1(R)$. (b) Let $\tilde{X}$ be a random element in $L^1(R)$. Then there exists a function $X : R \times \Omega \to R$ such that (i) for each $w \in \Omega, X(\cdot, w)$ is a Lebesgue integrable function, (ii) for each $t \in R, X(t, \cdot)$ is an extended random variable.

The expected value for a random element in a normed linear space is defined by the Pettis integral (cf. Taylor(1978)). In a separable Banach space, the Pettis integral is equal to the Bochner integral when the Bochner integral exists.

Lemma 2.2 (Araujo and Gine(1980). Let $(E, \| \ \|)$ be a separable Banach space and let X be a random element in E. Then X has a Bochner integral $EX \in E$ if and only if $E\|X\| < \infty$. In this case, $\|EX\| \leq E\|X\|$.

The following lemma gives a characterization of expected values in $L^1(R)$.

Lemma 2.3 (Lee(1990)) Let $\tilde{X}$ be a random element in $L^1(R)$ such that $E\|X\| < \infty$. Then there exists a unique $\tilde{EX} \in L^1(R)$ such that

$$\text{(i) } f(\tilde{EX}) = E[f(\tilde{X})] \quad \text{for every } f \in L^1(R)^*$$

and

$$\text{(ii) } \tilde{EX} = \tilde{E}[X(t, \cdot)].$$

Let $\{X_n\}$ be a sequence of random elements on a probability space (Ω, A, P) taking values in a separable normed linear space E and let $r > 0$. Then $\{X_n\}$ is said to be compact uniformly rth-order integrable if for every $\epsilon > 0$ there exists a compact subset K of E such that $\sup_n E\left[\left\|X_n I_{[X_n \in K^c]}\right\|^r\right] < \epsilon$. Compact uniform integrability denotes the case $r = 1$.

3. Convergence in Probability

Weak laws of large numbers (WLLN'S) are proved in this section. For a sequence of compact uniformly integrable random elements in $L^1(R)$, it is shown that pointwise convergence conditions are sufficient for WLLN's. For weighted sums of triangular arrays of random elements in $L^1(R)$, a WLLN is proved. These results are the $L^1(R)$ counterparts to the Daffer and Taylor(1979)'s results for $D[0,1]$. The following lemmas are needed in the proofs of the Theorems.

Lemma 3.1 Let K be a compact subset of $L^1(R)$. Then for each $\epsilon > 0$ there exists a constant m_k such that $\sup_{\tilde{x}\in K}\left\|\tilde{x} - \tilde{x}I_{[|t|\le m_k]}\right\| < \epsilon$.

Proof. Since K is compact, for given $\epsilon > 0$ there exist $\tilde{x}_1, \cdots, \tilde{x}_s \in K$ such that $\bigcup_{i=1}^{s}\{\tilde{y} : \|\tilde{y} - \tilde{x}_i\| < \epsilon/3\} \supset K$. Thus, for each $\tilde{x} \in K$ there exists $\tilde{x}_i$ such that $\|\tilde{x} - \tilde{x}_i\| < \epsilon/3$. Since $\tilde{x}_i \in L^1(R)$ implies $\left\|\tilde{x}_i - \tilde{x}_i I_{[|t|\le n]}\right\| \to 0$ as $n \to \infty$, we can choose m_k such that $\sup_{1\le i\le s}\left\|\tilde{x}_i - \tilde{x}_i I_{[|t|\le m_k]}\right\| < \epsilon/3$. Hence, for each $\tilde{x} \in K$

$$\begin{aligned}\left\|\tilde{x} - \tilde{x}I_{[|t|\le m_k]}\right\| &\le \|\tilde{x} - \tilde{x}_i\| + \left\|\tilde{x}_i - \tilde{x}_i I_{[|t|\le m_k]}\right\| \\ &\quad + \left\|\tilde{x}_i I_{[|t|\le m_k]} - \tilde{x}I_{[|t|\le m_k]}\right\| < \epsilon.\end{aligned}$$

Lemma 3.2 (Wang and Rao(1987)) Let $\{X_n\}$ be a sequence of random elements defined on a probability space (Ω, A, P) taking values in a separable Banach space E. Let $r > 0$. Then statements
(i) and (ii) are equivalent :

(i) $\{X_n\}$ is uniformly tight and $\{\|X_n\|^r\}$ is uniformly absolutely continuous.

(ii) $\{X_n\}$ is compact uniformly rth-order integrable.

Lemma 3.3 (Wang and Rao(1987)) Let $\{X_n\}$ be a sequence of random elements taking values in a separable Banach space E. If $\{X_n\}$ is compact uniformly rth-order integrable for some $r \ge 1$, then $\{X_n - EX_n\}$ is compact uniformly rth-order integrable.

Theorem 3.4 Let $\{\tilde{X}_{nk} : 1 \le k \le n, n \ge 1\}$ be a triangular array of random elements in $L^1(R)$ which are compact uniformly integrable with $\tilde{EX}_{nk} = \tilde{0}$ for each n and k and such that $|X_{nk}(t)| \le M$ for each t, k and n. Let $\{a_{nk}\}$ be an array of real numbers such that $\sum_{k=1}^{n}|a_{nk}| \le \Gamma < \infty$ for each n.
If $\sum_{k=1}^{n} a_{nk}X_{nk}(t) \to 0$ in probability for each $t \in R$, then
$E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}\right\| \to 0$.

Proof.
Let $\epsilon > 0$ be given. Since

$$E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}\right\| = \Gamma E\left\|\sum_{k=1}^{n}(a_{nk}/\Gamma)\tilde{X}_{nk}\right\|,$$

without loss of generality we can assume $\sum_{k=1}^{n}|a_{nk}| \leq 1$ for each n. Choose K compact, convex and symmetric with $\tilde{0} \in K$ such that

$$\sup_{n,k} E\left\|\tilde{X}_{nk}I_{[\tilde{X}_{nk}\in K]}\right\| < \epsilon/4. \tag{1}$$

By Lemma 3.1 pick a constant m_k such that

$$\sup_{\tilde{x}\in k}\left\|\tilde{x} - \tilde{x}I_{[|t|\leq m_k]}\right\| < \epsilon/4. \tag{2}$$

Thus, for each n

$$\begin{aligned} E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}I[|t| > m_k]\right\| &\leq E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}I_{[\tilde{X}_{nk}\in K]}I[|t| > m_k]\right\| \\ &\quad + E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}I_{[\tilde{X}_{nk}\notin K]}I[|t| > m_k]\right\| \\ &< \epsilon/4 + \sum_{k=1}^{n}|a_{nk}|\,E\left\|\tilde{X}_{nk}I_{[\tilde{X}_{nk}\notin K]}\right\| \\ &< \epsilon/4 + \epsilon/4 = \epsilon/2. \end{aligned} \tag{3}$$

from (1) and (2). Since $|\sum_{k=1}^{n} a_{nk}X_{nk}(t)| \to 0$ in probability and $|\sum_{k=1}^{n} a_{nk}X_{nk}(t)| \leq M$ for each $t \in R$, it follows that $E|\sum_{k=1}^{n} a_{nk}\ X_{nk}(t)| \leq M$ for each n and $E|\sum_{k=1}^{n} a_{nk}\ X_{nk}(t)| \to 0$ for each $t \in R$. Thus, by the bounded convergence theorem

$$E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}I_{[|t|\leq m_k]}\right\| = \int_{-m_k}^{m_k} E\left|\sum_{k=1}^{n} a_{nk}X_{nk}(t)\right|d\mu(t) \to 0 \tag{4}$$

as $n \to \infty$. Hence, from (3) and (4) there exists N such that for all $n \geq N$

$$E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}\right\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $E\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}\right\| \to 0$ implies $\left\|\sum_{k=1}^{n} a_{nk}\tilde{X}_{nk}\right\| \to 0$ in probability, several WLLN's follow as corollaries to Theorem 3.4. The following corollary follows from Lemma 3.3 and Theorem 3.4 by letting $a_{nk} = 1/n$ for $1 \leq k \leq n$.

Corollary 3.5 Let $\{\tilde{X}_k\}$ be a sequence of compact uniformly integrable random elements in $L^1(R)$ such that $\sup_{t,k} |X_k(t)| < \infty$. If

$$\frac{1}{n}\sum_{k=1}^{n}[X_k(t) - EX_k(t)] \to 0 \quad \textit{in probability}$$

for each $t \in R$, then

$$E\left\|\frac{1}{n}\sum_{k=1}^{n}(\tilde{X}_k - E\tilde{X}_k)\right\| \to 0.$$

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