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ESTIMATIONS OF THE GENERALIZED REIDEMEISTER NUMBERS II

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Abstract. This paper is a continuation of [1]. Let $\sigma(X, x_0, G)$ be the fundamental group of a transformation group (X, G). Let $R(\varphi, \psi)$ be the generalized Reidemeister number for an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$. The main results in this paper concern the conditions for $R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|$.

1. Introduction

F. Rhodes introduced the concept of the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G), a group G of homeomorphisms of a space X, as a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space X in [6]. Recently, we gave a definition of the generalized Reidemeister number $R(\varphi, \psi)$ for an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$ and studied the algebraic computations of $R(\varphi, \psi)$ in [1] and [5].

This article deals with the problem of determining the conditions for $R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|$ as a continuation of [1].

Throughout this paper, the space X is assumed to be a compact connected polyhedron. In this paper, we follow F. Rhodes [6] for the basic terminologies.

2. Preliminaries

Let (X, G) be a transformation group and let $(\varphi, \psi) : (X, G) \to (X, G)$ be an endomorphism. Since $\varphi(gx) = (\psi g)(\varphi x)$ for every pair (x, g), if α is a path in X of order g with base-point x_0, then $\varphi\alpha$ is a path in X of order $\psi(g)$ with base-point $\varphi(x_0)$. Furthermore, if two

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path α and β of the same order g is homotopic, $\alpha \simeq \beta$, then $\varphi\alpha \simeq \varphi\beta$. Thus (φ, ψ) induces a homomorphism

$$(\varphi, \psi)_* : \sigma(X, x_0, G) \to \sigma(X, \varphi(x_0), G)$$

defined by $(\varphi, \psi)_*[\alpha; g] = [\varphi\alpha; \psi(g)]$.

If λ is a path from $\varphi(x_0)$ to x_0, then λ induces an isomorphism

$$\lambda_* : \sigma(X, \varphi(x_0), G) \to \sigma(X, x_0, G)$$

defined by $\lambda_*[\alpha; g] = [\lambda\rho + \alpha + g\lambda; g]$ for each $[\alpha; g] \in \sigma(X, \varphi(x_0), G)$, where $\rho(t) = 1-t$. This isomorphism λ_* depends only on the homotopy class of λ.

Conveniently, we denote by $(\varphi, \psi)_\sigma$ the composition $\lambda_*(\varphi, \psi)_*$.

Definition. ([5]) Let $(\varphi, \psi)_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism. Two elements $[\alpha; g_1]$, $[\beta; g_2]$ in $\sigma(X, x_0, G)$ are said to be $(\varphi, \psi)_\sigma$*–equivalent*, $[\alpha; g_1] \sim [\beta; g_2]$, if there exists $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$[\alpha; g_1] = [\gamma; g][\beta; g_2](\varphi, \psi)_\sigma([\gamma; g]^{-1}).$$

For an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$, the *Reidemeister number* $R(\varphi, \psi)$ of (φ, ψ) is defined to be the numbers of equivalence classes of $\sigma(X, x_0, G)$ under $(\varphi, \psi)_\sigma$–equivalence.

3. The estimates of the generalized Reidemeister number

In this section, we always assume that the group G is an abelian.

Let $C(\sigma(X, x_0, G))$ be a commutator subgroup of $\sigma(X, x_0, G)$ and let

$$\bar{\sigma}(X, x_0, G) = \sigma(X, x_0, G)/C(\sigma(X, x_0, G)).$$

Then $\theta_\sigma : \sigma(X, x_0, G) \to \bar{\sigma}(X, x_0, G)$ is a canonical homomorphism such that $Ker\theta_\sigma = C(\sigma(X, x_0, G))$. Let $\eta_{\bar{\sigma}} : \bar{\sigma}(X, x_0, G) \to Coker(1 - (\varphi, \psi)_{\bar{\sigma}})$ be the natural projection. Then $\eta_{\bar{\sigma}}\theta_\sigma$ is an epimorphism.

Theorem 3.1. *([5]) If $(\varphi, \psi) : (X, G) \to (X, G)$ is an endomorphism, then $R(\varphi, \psi) \geq |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|$, where 1 and $(\varphi, \psi)_{\bar{\sigma}}$ denote respectively the identity isomorphism and the endomorphism of $\bar{\sigma}(X, x_0, G)$ induced by (φ, ψ). Furthermore, if $\sigma(X, x_0, G)$ is abelian,*

$$R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|.$$

THEOREM 3.2. *If the epimorphism $\eta_{\bar{\sigma}}\theta_\sigma$ induces a one-one correspondence between the set of $(\varphi, \psi)_\sigma$–equivalent classes and $Coker(1-(\varphi,\psi)_{\bar{\sigma}})$, then $[\alpha; g_1] \sim [\beta; g_2]$ implies $[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g]$ for any $[\gamma; g] \in \sigma(X, x_0, G)$.*

Proof. Note that the $\eta_{\bar{\sigma}}\theta_\sigma$ images of all elements of a $(\varphi, \psi)_\sigma$ – equivalent class are the same element of $Coker(1-(\varphi,\psi)_{\bar{\sigma}})$, that is, if $[\alpha; g_1] \sim [\beta; g_2]$, then $\eta_{\bar{\sigma}}\theta_\sigma([\alpha; g_1]) = \eta_{\bar{\sigma}}\theta_\sigma([\beta; g_2])$ (See proof of Theorem 3.5 in [5]). Since $\eta_{\bar{\sigma}}\theta_\sigma$ is a homomorphism,

$$\begin{aligned}\eta_{\bar{\sigma}}\theta_\sigma([\alpha; g_1][\gamma; g]) &= \eta_{\bar{\sigma}}\theta_\sigma([\alpha; g_1]) + \eta_{\bar{\sigma}}\theta_\sigma([\gamma; g])\\ &= \eta_{\bar{\sigma}}\theta_\sigma([\beta; g_2]) + \eta_{\bar{\sigma}}\theta_\sigma([\gamma; g])\\ &= \eta_{\bar{\sigma}}\theta_\sigma([\beta; g_2][\gamma; g]).\end{aligned}$$

Hence from the assumption of Theorem, we obtain

$$[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g].$$ □

COROLLARY 3.3. *If $(\varphi, \psi) : (X, G) \to (X, G)$ is an endomorphism, then the following statements are equivalent:*

(1) *The epimorphism $\eta_{\bar{\sigma}}\theta_\sigma$ induces a one-one correspondence between the set of $(\varphi, \psi)_\sigma$–equivalent classes and $Coker(1-(\varphi,\psi)_{\bar{\sigma}})$.*

(2) *For any $[\gamma; g] \in \sigma(X, x_0, G)$, $[\alpha; g_1] \sim [\beta; g_2]$ implies $[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g]$.*

(3) For any $[\alpha; g_1], [\beta; g_2], [\gamma; g_3] \in \sigma(X, x_0, G)$,

$$[\alpha; g_1][\beta; g_2][\gamma; g_3] \sim [\beta; g_2][\alpha; g_1][\gamma; g_3].$$

Proof. For (2) ⇒ (3) ⇒ (1), we refer to [1, Lemma 2.4 and Theorem 3.2]. Hence it is clear from Theorem 3.2. □

COROLLARY 3.4. *If one of the three statements in Corollary 3.3 holds, then*

$$R(\varphi, \psi) = |Coker(1-(\varphi,\psi)_{\bar{\sigma}})|.$$

Proof. From the first statement of Corollary 3.3, the proof is straight forward. □

LEMMA 3.5. *([1]) Let $(\varphi, \psi)_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism. Then, for any $[\alpha; g_1], [\beta; g_2] \in \sigma(X, x_0, G)$,*

(1) $[\alpha; g_1][\beta; g_2] \sim [\beta; g_2](\varphi, \psi)_\sigma([\alpha; g_1])$.
(2) $[\alpha; g_1] \sim (\varphi, \psi)_\sigma([\alpha; g_1])$.

THEOREM 3.6. *Let $Z(\sigma(X, x_0, G))$ be a center of $\sigma(X, x_0, G)$. If $(\varphi, \psi)_\sigma$ image of $\sigma(X, x_0, G)$ is contained in $Z(\sigma(X, x_0, G))$, that is,*

$$(\varphi, \psi)_\sigma(\sigma(X, x_0, G)) \subseteq Z(\sigma(X, x_0, G)),$$

then

$$R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|.$$

Proof. It is sufficient to prove that the third statement of Corollary 3.3 holds. For any $[\alpha; g_1], [\beta; g_2], [\gamma; g_3] \in \sigma(X, x_0, G)$, from (2) of Lemma 3.5 and hypothesis of Theorem,

$$\begin{aligned}
[\alpha; g_1][\beta; g_2][\gamma; g_3] &\sim (\varphi, \psi)_\sigma([\alpha; g_1][\beta; g_2][\gamma; g_3]) \\
&= (\varphi, \psi)_\sigma([\alpha; g_1])(\varphi, \psi)_\sigma([\beta; g_2])(\varphi, \psi)_\sigma([\gamma; g_3]) \\
&= (\varphi, \psi)_\sigma([\beta; g_2])(\varphi, \psi)_\sigma([\alpha; g_1])(\varphi, \psi)_\sigma([\gamma; g_3]) \\
&= (\varphi, \psi)_\sigma([\beta; g_2][\alpha; g_1])(\varphi, \psi)_\sigma([\gamma; g_3]) \\
&= (\varphi, \psi)_\sigma([\beta; g_2][\alpha; g_1][\gamma; g_3]) \\
&\sim [\beta; g_2][\alpha; g_1][\gamma; g_3].
\end{aligned}$$

□

COROLLARY 3.7. *([1]) Let $(\varphi, \psi)_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism. If $\sigma(X, x_0, G)$ is abelian, then*

$$R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|.$$

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