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THE GLOBAL BEHAVIORS OF A HOPF BIFURCATION IN A FREE BOUNDARY PROBLEM WITH ZERO FLUX BOUNDARY CONDITION

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Abstract. The local behaviors of the Hopf bifurcation in the free boundary problem satisfying the zero flux boundary condition was examined in [3]. In this paper, we shall examine the global behaviors for this problem and shall apply the center-index theory to show the globality.

1. Introduction

The propagator controller system with the McKean reaction term is reduced to a free boundary problem when the layer parameter is equal to zero. In [3], the authors showed the local existence of a Hopf bifurcation of the problem satisfying the zero flux boundary condition:

$$
(1.1) \qquad \begin{cases} v_t = Dv_{xx} - c^2 v + H(x - s(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+, \\ v(0,t) = 0 = v(1,t) & \text{for } t > 0, \\ v(x,0) = v_0(x) & \text{for } 0 \le x \le 1, \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0, \\ s(0) = s_0, & \end{cases}
$$

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where $v(x,t)$ and $v_x(x,t)$ are required to be continuous in Ω. Here $\Omega = (0,1)\times(0,\infty)$, $\Omega^- = \{(x,t)\in\Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t)\in\Omega : s(t) < x < 1\}$. Furthermore, τ is the bifurcation parameter and $H(y)$ denotes the Heaviside unit step function.

The local existence of a Hopf bifurcation means that at a critical value τ^* of τ, the stationary solution $(v^*(x), s^*)$, $1/3 < s^* < 1$ with $0 < 1 - 2a < 2/c^2$ loses stability and a branch of stable periodic solutions appears for a finite diffusion constant D. Moreover, the steady state is stable for $\tau > \tau^*$ and unstable for $\tau < \tau^*$, and τ^* is a bifurcation point for a stable branch of periodic orbits which turns in the direction $\tau < \tau^*$. This Hopf bifurcation guarantees the existence of small amplitude, nontrivial periodic curve bifurcating from the Hopf point $(v^*(x), s^*, \tau^*)$. In this paper, we shall examine the global behaviors of the Hopf bifurcation and shall investigate the existence of a bifurcating continuum of periodic orbits containing another center which is not a Hopf point. Here, a center means that some eigenvalues of the linear part of (1.1) are purely imaginary and not zero. In order to do this, we need to recall the regularization of the system (1.1) from [3]:

$$\text{(R)} \qquad \begin{cases} \dfrac{du}{dt} + Au = \dfrac{1}{\tau}G(x,s)C(u(s)+\gamma(s)) \\ s'(t) = \frac{1}{\tau}C(u(s)+\gamma(s)) \\ u(0) = u_0, \;\; s(0) = s_0. \end{cases}$$

Here $u(t)(x) := v(x,t) - g(x,s(t))$ where $g(x,s) := \int_s^1 G(x,y)\,dy = A^{-1}(H(\cdot - s))(x)$ and $\gamma(s) := g(s,s)$.

In section 2, we shall investigate the behaviors of the real part of eigenvalues for the problem (R) and show the globality of the Hopf bifurcation in the last section.

2. The properties of real eigenvalues

In this section, we shall examine the properties of the real part of eigenvalues for the problem (R).

The problem (1.1) has the uniquely determined stationary solution $(v^*(x), s^*)$, $1/3 < s^* < 1$ if $0 < 1 - 2a < 2/c^2$. In [3], the linearized eigenvalue problem for (1.1) is given by

$$(2.1) \qquad \begin{cases} (A+\lambda)v &= -\delta_{s^*} \\ \rho\cdot\lambda &= \gamma'(s^*) + G(s^*,s^*) + v(s^*), \end{cases}$$

where δ_{s^*} is the Dirac delta function and $\rho = \tau/4$.

We define the set $S_\nu := \{\lambda \in C \,|\, \operatorname{Re}\lambda > -c^2\}$. In the first equation of (2.1), $A + \lambda$ is invertible in S_{c^2} and hence it has a unique solution $v = -G_\lambda(\cdot, s^*)$, where G_λ is a Green's function for the operator $A + \lambda$. It follows that the second equation of (2.1) can be written as

$$\rho \cdot \lambda = \gamma'(s^*) + G(s^*, s^*) - G_\lambda(s^*, s^*). \tag{2.2}$$

We denote $\alpha = \operatorname{Re}\lambda$ and $\beta = \operatorname{Im}\lambda$ where $\operatorname{Re}\lambda$ is the real part of the eigenvalue and $\operatorname{Im}\lambda$ is the imaginary part of the eigenvalue. In the next two lemmas. we show some properties of the function $G_\lambda(s^*, s^*)$:

Lemma 2.1. *The function $G_\alpha(s^*, s^*)$ is a strictly decreasing convex function of α, $\alpha > -c^2$, and*

$$\lim_{\alpha \to -c^2} G_\alpha(s^*, s^*) = s^*(1 - s^*), \qquad \lim_{\alpha \to \infty} G_\alpha(s^*, s^*) = 0\,.$$

Furthermore, $\dfrac{dG_\lambda}{d\lambda}(s^*, s^*) \neq 0$ *for those values of λ with* $\operatorname{Im}\lambda \neq 0$.

Proof. Since the operator $(A+\lambda)^{-1}$ exists for $\operatorname{Re}\lambda > -c^2$, $\lim_{\alpha\to\infty}(A+\alpha)^{-1} = 0$ and thus we obtain $\lim_{\alpha\to\infty} G_\alpha(s^*, s^*) = 0$. The function $G_\alpha(s^*, s^*)$ is represented by

$$G_\alpha(s^*, s^*) = \frac{\sinh(s^*\sqrt{\alpha^2 + c^2})\,\sinh((1 - s^*)\sqrt{\alpha^2 + c^2})}{\sqrt{\alpha^2 + c^2}\,\sinh(\sqrt{\alpha^2 + c^2})}$$

and hence we have

$$\lim_{\alpha \to -c^2} G_\alpha(s^*, s^*) = s^*(1 - s^*).$$

In order to show that $\alpha \mapsto G_\alpha(s^*, s^*)$ is a strictly decreasing function, we define $h(\lambda)(x) := G_\lambda(x, s^*) - G(x, s^*)$. Then (in the weak sense at first)

$$(A + \lambda)h(\lambda) = -\lambda G(\cdot, s^*)\,.$$

It follows that $h(\lambda) \in D(A)$ and $h : \mathbf{R}^+ \to D(A)$ is differentiable with

$$(A + \lambda)h'(\lambda) = -G_\lambda(\cdot, s^*)\,.$$

Multiplying by $(A + \lambda)\overline{h'(\lambda)}$ and integrating both sides, we obtain the real and imaginary parts

$$\int_0^1 \left(|Ah'(\lambda)|^2 + (\alpha^2 - \beta^2)|h'(\lambda)|^2 + 2\alpha A|h'(\lambda)|^2\right) dx = -\operatorname{Re}\left(h'(\lambda)(s^*)\right),$$

$$2\beta\Big(\int_0^1 (A+\alpha)|h'(\lambda)|^2\,dx\Big) = \mathrm{Im}\,(h'(\lambda)(s^*)). \tag{2.3}$$

For $\beta = 0$, equation (2.3) becomes

$$\int_0^1 |(A+\alpha)h'(\alpha)|^2\,dx = -h'(\alpha)(s^*) > 0.$$

From the definition of h, we have $h'(\lambda)(s^*) = \dfrac{dG_\lambda}{d\lambda}(s^*, s^*)$, which implies that G_α is a strictly decreasing function of α. Moreover, we obtain that $\mathrm{Im}\Big(\dfrac{dG_\lambda}{d\lambda}\Big) \neq 0$ holds if and only if $\beta \neq 0$ as follows from (2.3).

Finally, we show the convexity of G_α. Differentiate the equation $h(\alpha)+(A+\alpha)h'(\alpha) = -G(\cdot, s^*)$ with respect to α and then multiply $(A+\alpha)^2 h''(\alpha)$ by $h''(\alpha)$ and integrate both sides. Then we obtain

$$\begin{aligned}
\int_0^1 (A+\alpha)^3\ h''(\alpha)^2\,dx &= -2\int_0^1 (A+\alpha)^2 h'(\alpha)h''(\alpha)\,dx \\
&= -2\int_0^1 (A+\alpha)(-G_\alpha(x,s^*))h''(\alpha)\,dx \\
&= 2h''(\alpha)(s^*).
\end{aligned}$$

Since $h''(\alpha) = \dfrac{d^2G_\alpha}{d\alpha^2}(s^*, s^*)$, the convexity of G_α is shown. □

Lemma 2.2. *For some negative number $-\hat{\lambda}$, the function $\dfrac{dG_\lambda}{d\lambda}(s^*, s^*)$ evaluated at complex eigenvalues has the following property*

$$\begin{aligned}
-\frac{dG_\lambda}{d\lambda}(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=-\hat{\lambda},\mathrm{Im}\lambda=0)} &> -\frac{dG_\lambda}{d\lambda}(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=0,\mathrm{Im}\lambda=0)} \\
&> -\tfrac{1}{\beta}\,\mathrm{Im}\,G_\beta(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=0,\mathrm{Im}\lambda=\beta)}.
\end{aligned}$$

Proof. We use the Fourier sine representation of G_λ and let $-\hat{\lambda}$ be a negative constant and $\beta \neq 0$. Differenting G_λ with respect to λ and

evaluating at $\mathrm{Re}\,\lambda = -\hat{\lambda}$ and $\mathrm{Im}\,\lambda = 0$, one obtains

$$
\begin{aligned}
-\frac{dG_\lambda}{d\lambda}(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=-\hat{\lambda},\mathrm{Im}\lambda=0)} &= 2\sum_{k=1}^{\infty} \frac{(\sin k\pi s^*)^2}{(k^2\pi^2 + c^2 - \hat{\lambda})^2} \\
&> 2\sum_{k=1}^{\infty} \frac{(\sin k\pi s^*)^2}{(k^2\pi^2 + c^2)^2} = -\frac{dG_\lambda}{d\lambda}(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=0,\mathrm{Im}\lambda=0)} \\
&> 2\sum_{k=1}^{\infty} \frac{(\sin k\pi s^*)^2}{(k^2\pi^2 + c^2)^2 + \beta^2} = -\frac{1}{\beta}\,\mathrm{Im}G_\beta(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=0,\mathrm{Im}\lambda=\beta)},
\end{aligned}
$$

where $G_\beta(s^*, s^*)$ is the Green's function of the operator of $A+i\beta$. Therefore, the lemma is shown. □

From (2.2), a real eigenvalue $\lambda = \alpha$ satisfies the equation

$$\gamma'(s^*) + G(s^*, s^*) - \rho\alpha = G_\alpha(s^*, s^*)\,, \tag{2.4}$$

where $\gamma'(s^*) + G(s^*, s^*)$ is a positive constant. The real eigenvalues of (2.4) can be determined by the locating the intersection of the curve $G_\alpha(s^*, s^*)$ with the straight line $\gamma'(s^*) + G(s^*, s^*) - \rho\alpha$ (see Figure 1).

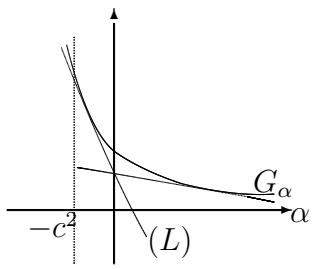

Figure 1: The graphs of G_α and $(L) : \gamma'(s^*) + G(s^*, s^*) - \rho\alpha$.

Let ρ_n be defined by

$$
\begin{aligned}
\rho_n := \min\{\ \rho \in \mathbf{R}\ :\ & \text{there exists at least one negative real eigenvalue} \\
& \text{between the line } \gamma'(s^*) + G(s^*, s^*) - \rho\alpha \text{ and } G_\alpha \\
& \text{for } \alpha > -c^2\}.
\end{aligned}
$$

We obtain the next lemma from a simple geometrical analysis.

Lemma 2.3. *There exists a positive constant ρ_T in S_{c^2} with $\rho_T < \rho_n$:*

(i) *there are no real eigenvalues of (2.4) for $\rho_T < \rho < \rho_n$*

(ii) *there exists a unique real positive eigenvalue λ_T at $\rho = \rho_T$*

(iii) *there exist exactly two real eigenvalues* $\lambda_1(\rho)$ *and* $\lambda_2(\rho)$ *for* $\rho < \rho_T$, *where* $-\rho_T$ *is the slope of the line which is tangent to the curve* $G_\alpha(s^*, s^*)$.

When ρ is close to ρ_T in the right hand side, the real eigenvalues are expected to be changed to complex eigenvalues. The local behavior of eigenvalues near (α_T, ρ_T) is described as follows;

LEMMA 2.4. *The positive real eigenvalue* α_T *corresponding to* $\rho = \rho_T$ *is of multiplicity two. Near* $\rho = \rho_T$, α_T *splits into two eigenvalues, since*

$$\begin{aligned} \lambda &\simeq \alpha_T \pm i\sqrt{\Delta_T(\rho - \rho_T)} \text{ for } \rho > \rho_T\,, \\ \lambda &\simeq \alpha_T \pm \sqrt{\Delta_T(\rho_T - \rho)} \text{ for } \rho < \rho_T \end{aligned}$$

with $\Delta_T = \left.\dfrac{2\alpha_T}{\frac{d^2G_\lambda}{d\lambda^2}}(s^*, s^*)\right|_{(\mathrm{Re}\lambda=\alpha_T,\, \mathrm{Im}\lambda=0)}$.

Proof. We define

$$F(\lambda, \rho) := \lambda\rho - (\gamma'(s^*) + G(s^*, s^*)) + G_\alpha(s^*, s^*).$$

Since for $\rho < \rho_n$, $\frac{\partial F}{\partial \lambda}(\lambda^*, \rho) = 0$ holds if and only if $(\lambda^*, \rho) = (\alpha_T, \rho_T)$, we obtain

$$F(\lambda^*, \rho) \simeq \alpha_T(\rho - \rho_T) + \frac{(\lambda - \alpha_T)^2}{2} \cdot \left.\frac{d^2G_\lambda}{d\lambda^2}(s^*, s^*)\right|_{(\mathrm{Re}\lambda=\alpha_T, \mathrm{Im}\lambda=0)}$$

by the Talor expansion. The conclusion follows from the above equation. □

Since the Hopf bifurcation occurred at $\rho = \rho^*$, the critical point ρ^* must lie in the interval (ρ_T, ρ_n). In the following lemma, we determine a subinterval of (ρ_T, ρ_n) containing ρ^*.

LEMMA 2.5. *There exist a positive constant* ρ_s *and* $\hat{\lambda}$ *such that there are no eigenvalues in* $S_{\hat{\lambda}}$ *for* $\rho_n > \rho \geq \rho_s$.

Proof. Let ρ_0 be the slope of $-G_\lambda(s^*, s^*)$ at $\mathrm{Re}\lambda = 0$, i.e.,

$$\rho_0 = -\left.\frac{dG_\lambda}{d\lambda}(s^*, s^*)\right|_{\mathrm{Re}\lambda=0}.$$

Then $\rho_n > \rho_0$. Let $\rho_n > \rho_s > \rho_0$. For $\rho_n > \rho \geq \rho_s$, there is no real eigenvalue. Thus we need to show that there is no complex eigenvalue for $\rho_n > \rho \geq \rho_s$, where $-\hat{\lambda}$ is a negative constant determined by

$$\rho_s = -\frac{dG_\lambda}{d\lambda}(s^*, s^*)\Big|_{\mathrm{Re}\lambda=-\hat{\lambda},\, \mathrm{Im}\lambda=0}.$$

By Lemma 2.2, we see that if $\mathrm{Re}\lambda > -\hat{\lambda}$ and $\mathrm{Im}\lambda > 0$, then

$$\rho_s > -\frac{dG_\lambda}{d\lambda}(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=0,\mathrm{Im}\lambda=0)} > -\frac{1}{\beta}\,\mathrm{Im}\, G_\beta(s^*, s^*)\Big|_{(\mathrm{Re}\lambda=0,\mathrm{Im}\lambda=\beta)},$$

which implies that there is no complex eigenvalue in $S_{\hat{\lambda}}$ since $\mathrm{Im}\, G_\beta(s^*, s^*) + \rho \cdot \beta = 0$ has a solution ρ^* when $\rho^* \geq \rho_s$ (see [3]). □

Remark 2.6. We note that $\rho_s > \rho_T$ and $\rho_s > \rho^*$.

3. The behaviors of the Hopf bifurcation

The behaviors of the complex eigenvalues with respect to ρ after crossing the imaginary axis will be described in this section. From Lemma 2.3, there is a unique real positive eigenvalue at $\rho = \rho_T$ and a pair of complex conjugate eigenvalues appears for $\rho > \rho_T$. At this stage, there may exist other complex eigenvalues, however, we can avoid such an existence in the following sense.

We now trace the behavior of these other complex eigenvalues as ρ increases from ρ_T. Since there is no real eigenvalue for $\rho_n > \rho > \rho_T$, they remain as complex eigenvalues and can be uniquely expressed as functions of ρ. By Lemma 2.4, they must cross the imaginary axis from right to left at some point $\rho = \hat{\rho}$ before ρ reaches ρ_s. However, because of the uniqueness of pure imaginary eigenvalues, $\hat{\rho}$ must be equal to ρ^* and the corresponding eigenvalues must be $\lambda(\rho^*)$. This establishes the global behavior of Hopf critical eigenvalues with respect to ρ. Therefore, we have the following theorem.

Theorem 3.1. *Suppose that* $0 < \dfrac{1-2a}{2} < \dfrac{1}{c^2}$. *Then we have :*

(i) *At* $\rho = \rho^*$, *all other eigenvalues lie strictly in the left half-plane in* $\mathbf{C}$.

(ii) *Following the Hopf bifurcation, the pure imaginary eigenvalues behave as follows:* $\lambda(\rho)$ *and* $\overline{\lambda(\rho)}$ *combine to make a real eigenvalue* α_T *of multiplicity two at* $\rho = \rho_T (< \rho^*)$*, which then it splits into the two real eigenvalues, say* $\lambda_1(\rho)$ *and* $\lambda_2(\rho)$ *for* $\rho < \rho_T$*. Moreover, for* $\rho \leq \rho^*$*, there is no eigenvalue except for those constructed above with some constant* $\hat{\lambda}$*.*

Now, in order to show the global Hopf bifurcation, we shall use the center index Φ introduced by Mallet-Paret & Yorke [2]. Let $E(\rho)$ denote the sum of the multiplicities of the eigenvalues of the linearization of (R) having strictly positive real parts. Let $E(\hat{\rho}+)$ an d $E(\hat{\rho}-)$ denote right- and left-hand limits of E at $\hat{\rho}$. Define the crossing number χ, the net number of pairs of eigenvalues crossing the imaginary axis at $\hat{\rho}$ by

$$\chi = \frac{1}{2}\Big(E(\hat{\rho}+) - E(\hat{\rho}-)\Big).$$

We define the center index of a center $(\hat{u}, \hat{s}, \hat{\rho})$ to be the product

$$\Phi(\hat{u}, \hat{s}, \hat{\rho}) = \chi \cdot (-1)^{E(\hat{\rho})}.$$

Essentially, a nonzero H-index

$$H := \Sigma\, \Phi \quad \neq 0$$

implies the global Hopf bifurcation, see [1]. Therefore, we must show that H-index is not zero.

Because of the global behavior of Hopf critical eigenvalues in Theorem 3.1, the Hopf point $(0, s^*, \rho^*)$ is the only center of (R). Thus, $E(\rho^*) = 0$ and $\chi = 1$ imply that a center index at $(0, s^*, \rho^*)$ is equal to 1. Hence, the H-index, $H = \Sigma\, \Phi = 1 \neq 0$. Therefore, we now have a global Hopf bifurcation.

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