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Nontrivial Complex Equivariant Vector Bundles over S1

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원위에서의 Nontrivial Complex Equivariant Vector Bundle

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Every complex vector bundle over S1 splits sum of line bundle and the first Chern class classify complex line bundle. This implies every complex vector bundle over S1 is trivial. In this paper, we show the existence of some nontrivial complex vector bundle over S1 in the equivariant case.

원위에서의 모든 복소 vector bundle**은** line bundle**로 나누어지며 첫째** Chern class**는 복소** line bundle**을 분류한다**. **이것은 원위에서의 모든 복소** vector bundle**은** trivial **하다는 것을 의미한다**. **이 논문에서는 군작용이 있을 경우에는 원위에서의 복소** vector bundle**중에** trivial**하지 않는** bundle**이 존재함을 보였다**.

Key words : complex vector bundle, equivariant vector bundle, Chern class, trivial vector bundle, nontrivial line bundle.

I. Nonequivariant case

1. Complex bundle case.

Every complex vector bundle over S1 splits sum of line bundles and the first Chern class classify complex line bundle. This implies every complex vector bundle over S1 is trivial. i.e., isomorphic to a product bundle S1 $\times$F, for some vector space F. We can see this fact from different points of view. Since every vector bundle is classified by classifying map and [S1, Bu(m)] $\cong$ π1(BU(m)) = π0 (U(m)) = {1}, it is easy to see that every vector bundle over S1 is trivial.

2. Real bundle case.

Every real vector bundle over S1 splits sum of line bundles and the first Stiefel-Whitney class classify real line bundle. This implies every real vector bundle over S1 is not trivial.

We can see this fact from different points of view. Since every vector bundle is classified by classifying map and $[S^1, BO(m)] \cong \pi_1(BO(m)) = \pi_0(O(m)) = Z_2 = \{\pm 1\}$ there are, up to isomorphism, only two vector bundles over S^1, the trivial bundle and the twist bundle. So it is easy to see that every vector bundle over S^1 is not trivial. Espectially, when m =1, one is a trivial line bundle and the other one is a Hopf line bundle $S^1 \times_{Z_2} R \to S^1$, where $Z_2 = \{\pm 1\}$.

II. Equivariant case.

Let G be a compact Lie group and let S^1 denote the unit circle in $\mathbf{R}^2$ with the standard metric. Since every smooth compact Lie group action on S^1 is smoothly equivalent to a unique linear action [See 5. TH 2.0.], we may think of S^1 with a smooth G-action as S(V) the unit circle of a real 2-dimentional orthogonal G-module V. In this section we consider smooth G-vector bundle over S(V).

In [3], we proved

Proposition. A smooth G-line bundle $L \to S^1$ is equivariantly isomorphic to a product bundle $S(V) \times \delta \to S(V)$ or $S(V) \times_{z_2} \delta \to S(V)/Z_2 = P(V)$ according as the G-line bundle $L \to S^1$ is trivial or not when we forget the action. Here S(V) denotes the unit circle of a real 2-dimensional orthogonal G-module V, δ a real 1-dimensional G-module and Z_2 acts on S(V) and δ as scalar multiplication.

In [4], we obtaine a similar result for a higher dimensional smooth G-vector bundle over S(V) when the G-action on S(V) is injective, in other words, when V is a faithful representation. In the non-equivariant case, real smooth vector bundles over S^1 are classified by the first Stiefel Whitney class. So there are, up to isomorphism, only two vector bundles over S^1, the trivial bundle and the twist bundle. In the equivariant case, we obtain the following results.

Theorem A. A smooth G-vector bundle over S^1 is equivariantly isomorphic to a Whitney sum of G-line bundle if the induced G-action on the base space is effective.

Theorem B. If the induced G-action on the base space is effective, then a smooth G-vector bundle $E \to S^1$ is equivariantly isomorphic to a product bundle $S(V) \times W^+ \to S(V)$ or a twist bundle $S(V) \times_{Z_2} (W^+ \oplus W^-) \to S(V)/Z_2 = P(V)$. Here S(V) denotes the unit circle of a real 2-dimensional orthogonal G-module V, W^+ and W^- is a real G-module and Z_2 acts on W^+ trivially and acts on W^- and S(V) as scalar multiplication.

Proof of Theorem A. Case 1. Suppose G is a subgroup of SO(2). Then G is SO(2) or cyclic group.

So G acts freely on the base space. In this case we can reduce our case to non-equivariant case by the bijection map $Vect_G(S^1) \to Vect(S^1/G)$ defined by $E \to E/G$

and $\xi \to \pi^*\xi$ [See 1. TH 1.6.1]

If G is SO(2), $Vect(S^1/G) = Vect(pt)$ which is trivial vector bundle in each dimension. So it is a whitney sum of trivial line bundle. If G is a cyclic group, $Vect(S^1/G) = Vect(S^1)$ which is the trivial bundle or the twist bundle. The twist bundle is a sum of trivial line bundles and Hopf line bundle. By taking pullback $\pi^*\xi$ of the above bundles ξ we get a whitney sum of G-line bundle over S^1.

Case 2. Suppose G is not a subgroup of SO(2). Then G is a Dihedral group D_n or O(2). Take a normal subgroup $N = G \cap SO(2)$ of G. Then N is SO(2) or a cyclic group. If N is SO(2), then N acts freely on the base space. So we get a bijection map $Vect_G(S^1) \to Vect_{Z_2}$

(S1/SO(2)) = VectZ2(pt) = Z2-representation. Since any Z2-representation is a sum of one dimensional Z2-representation, we get a whitney sum of G-line bundle by taking pullbuck of one dimensional Z2-representation. If N is cyclic group, we get a bijection map VectG(S1) $\rightarrow$ VectZ2(S1/N)=VectZ2(S1), where G acts on S1 by ρ : G $\rightarrow$ O(2) and any Z2 acts on S1 by reflextion. So it suffices to prove the following Lemma i.e., when G is Z2. Then by taking pullback we can conclude Theorem A.

Lemma. Suppose Z2 acts on S1 by reflextion. A Z2-vector bundle E over S1 is isomorphic to a whitney sum of Z2-line bundle.

Proof. Let {z0, z1} be the fixed set of Z2 on S1. Choose an eigenvector vi at zi and connect v0 and v1 by using a path to get a vector field on the upper half circle. Extend this vector field to the lower half circle by using Z2-action. Then we get a vector field on S1. This vector field may not be continuous. But each vector generate a line whose union is a Z2-line bundle. So we get a Z2-line bundle L1 over S1 which is a subbundle of E. So we can decompose En as follows:

En $\cong$ E1n-1 $\oplus$ L1.

we continue the above process until we get En $\cong$ L1 $\oplus$ L2 $\oplus$ $\cdots$ $\oplus$ Ln.

Proof of Theorem B.

By Theorem A, suppose our G-vector bundle E is a Whitney sum of trivial G-line bundle then E $\cong$ S(V) $\times$W+, where W+= δ1 $\oplus$ $\cdots$ $\oplus$ δn and δ1 is a real 1-dimensional G-module. If E is a Whitney sum of twist G-line bundle, then E $\cong$S(V) $\times$ Z2W-, where W-= δ1 $\oplus$ $\cdots$ $\oplus$ δn and δ1 is a real 1-dimensional G-module and Z2 acts on δ1 as a scalar multiplication. In general, we obtain the following results:

E $\cong$ P(V) $\times$W+ $\oplus$ S(V) $\times$ Z2W-

= S(V) $\times$ Z2 W+ $\oplus$ S(V) $\times$ Z2W-

= S(V) $\times_{Z2}$ (W+ $\oplus$ W-), where P(V)= S(V)/ Z2.

III. Example of nontrivial equivarinat complex line bundle.

In section I, we mentioned that every complex vector bundle over S1 is trivial (i.e., isomorphic to product bundle) if G is trivial. In this section we will show the existence of nontrivial G-line bundle over S1 by example. Let G be a compact Lie group and let V be a real 2-dimensional orthogonal G-module. We denote the representa -tion associated with V by ρ : G $\rightarrow$ O(2) and the unit circle of V by S(V). Note that effectiveness of the G-action is equivalent to the injectivity of ρ. If G is abelian, ρ is an abelian subgroup of O(2); so it is contained in SO(2) or isomorphic to D1 or D2, where Dn denotes the dihedral subgroup of O(2) generated by the reflection matrix with respect to the x-axis and the rotation matrix of angle 2 π/n. Without loss of generality we may assume that ρ agrees with D1 or D2 unless it is contained in SO(2).

EXAMPLE. Suppose that ρ =D1. Take a complex 1-dimensional G-module M and denote the associated representation by μ: G $\rightarrow$ GL(1,C)=**C***. We define a G-action on S1 $\times$ **C** by

$$g(z,v)=\begin{cases}(z,\ \mu(g)v) & \text{if } g \in \ker \rho,\\ (\bar{z},\ \mu(g)\bar{z}v) & \text{if } g \notin \ker \rho,\end{cases}$$

where (z,v) $\in$S1 $\times$ **C** and S1 denotes the unit circle of **C**. The projection onto the first factor makes it a complex G-vector bundle over S(V), which we denote by $\underline{M}$. One can check that the fiber representations at the two fixed points in S(V) are different, in fact, they are isomorphic to M and M $\otimes$ c δ, where δ

denotes the nontrivial 1-dimensional G-module with ker ρ acting trivially.

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