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ON THE DEFECTS AND TEXTURES IN THE ORDERED MEDIUM

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Abstract We find the properties of the defects and textures in an ordered medium. Especially, the space X, e.g., the residually solvable space or space satisfying the conditions (T^{**}) with respect to the defect and texture.

1. Introduction

The study of the defects and textures in an ordered medium was proceeded partially during twenty years. Concretely, N. S. Mermin [9], V. Peonaru and G. Toulouse have been studied an order medium from the nematic liquid crystal state or three dimensional spin state or Klein bottle respectively. By use of the homotopical method they classified the defects and textures in an ordered medium.

We work in the category of the topological spaces having the homotopy type of connected CW-complexes with a base point if the topological space is concerned.

2. Some properties of the defects and textures in an ordered medium

We say that a space $X(\in T)$ satisfies the condition (T^*) [4] if for all $g, t \in \pi_1(X)$ either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$.

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We defined the following condition (T^{**}) and controlled the locally nilpotent space effectively as follows [5, 6].

For $X \in T$, we say that X satisfies the condition (T^{**}) if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$.

We will study more deeply how homotopy theory has been used to classify defects and textures in an ordered medium. Before we start to check the concept of the ordered medium, first of all we consider the case in the two dimensional space.

In this case the ordered medium is a region of R^2 with the property that at any point there is a spin vector $\overrightarrow{S}$. The vector $\overrightarrow{S}$ has a fixed length, is free to a point in any direction in R^2 and is a continuous function of $\overrightarrow{r} \in R^2$, except possibly for a set $\Sigma(\subset R^2)$ which is called a defect set (could, of course, be empty). For example, we can write, for $\overrightarrow{r} \in R^2 - \Sigma$

$\overrightarrow{S}(\overrightarrow{r}) = \hat{u}cos\phi(\overrightarrow{r}) + \hat{v}sin\phi(\overrightarrow{r})$,

where $\hat{u}, \hat{v}$ are a fixed pair of orthonormal vectors and $\phi : R^2 \to R$ is a continuous map.

The ordered medium will be said to be in a uniform state if $\overrightarrow{S}(\overrightarrow{r})$ is independent of $\overrightarrow{r}$. The defect set Σ for the planar spin system need not be empty. In fact, topologically stable point defects are possible for this system. From the two dimensional case above, we can derive the following definitions of a defect and an ordered medium.

DEFINITION 2.1. Ordered medium is a kind of topological space V called the one parameter space.

DEFINITION 2.2. In X, there is a subset $\Sigma \subset X$ which will be called the set of defects. Outside Σ a continuous map ϕ is called the order parameter vector field. We call the subset Σ the set of defect of X in $\phi : X - \Sigma \to V$.

The physical space M^3 will be taken to be a smooth 3-dimension al manifold and M^3 is assumed to be connected, orientable and compact with boundary ∂M^3.

Now let's turn to the concept of the texture. We think the texture only in a 3-dimensional case. Consider a 3-dimensional ordered medium with order parameter V and suppose that the order medium is uniform at points far from the origin. A configuration of this ordered medium can be regarded as a mapping from a 3-dimensional cubical box I^3 into a single point in V. But such a map corresponds to an element of $\pi_3(V)$. If $\pi_3(V) = 0$ then any two configurations of the ordered medium are topologically identitical as can be shown using essentially the arguments [9].

DEFINITION 2.3. If $\pi_3(V) \neq 0$ then we have the interesting possibility of topologically inequivalent nonsingular configuration in an ordered medium. These are called textures.

We recall the following [8].

DEFINITION 2.4. A fibration $F \to E \to B$ is said to be quasi-nilpotent if the action of $\pi_1(B)$ on $H_n(F)$ is nilpotent, $n \geq 0$. Furthermore the fibration $F \to E \to B$ is strong quasi-nilpotent if it is quasi-nilpotent and if, in addition, $\pi_1(B)$ is nilpotent.

We recall that a group G has the property χ residually if to every element $g(\neq 1) \in G$, there is a normal subgroup N of G such that $g \notin N$ and G/N has the property χ [10].

Let's say that a group action G on H is solvable if there exists a finite chain; $H = H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_j \supset \cdots \supset H_n = \{e\}$ such that for each j

(1) H_j is closed under the action of G,
(2) H_{j+1} is normal in H_j and H_j/H_{j+1} is abelian.

DEFINITION 2.5. We define that a space $X(\in T)$ is (residually) solvable if

(1) $\pi_1(X)$ is (residually) solvable, and
(2) there is a (residually) solvable action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ for all $n \geq 2$.

And the category of (residually) solvable spaces and continuous maps is denoted by $(T_{RS}), T_S$ respectively. The category T_S has a finite product property from the definition of T_S;

THEOREM 2.6. *For a set* $\{X_\alpha | \alpha \in M : \textit{finite}\}$, $X_\alpha \in T_S$ *for any* α *if and only if* $\prod_{\alpha \in M} X_\alpha \in T_S$.

Similarly, the category T_{RS} also has a finite product property:

THEOREM 2.7. *For a set* $\{X_\alpha | \alpha \in M : \text{finite}\}$, $X_\alpha \in T_{RS}$ *for any* α *if and only if* $\prod_{\alpha \in M} X_\alpha \in T_{RS}$.

Proof. First, we only prove the following: for any $X_\alpha, X_\beta \in \{X_\alpha | \alpha \in M : \text{finite}\}$ such that $X_\alpha, X_\beta \in T_{RS}$, then $X_\alpha \times X_\beta \in T_{RS}$.

Since X_α and X_β are residually solvable spaces, for any element $g_\alpha(\neq 1) \in \pi_1(X_\alpha), g_\beta(\neq 1) \in \pi_1(X_\beta)$, there exist the nontrivial normal subgroups $(g_\alpha \notin)H_\alpha$ and $(g_\beta \notin)H_\beta$ in $\pi_1(X_\alpha)$ and $\pi_1(X_\beta)$, such that $\pi_1(X_\alpha)/H_\alpha$ and $\pi_1(X_\beta)/H_\beta$ are solvable groups respectively. Thus to any element $(g_\alpha, g_\beta)(\neq 1)$ as above we have a nontrivial normal subgroup $((g_\alpha, g_\beta) \notin)H_\alpha \oplus H_\beta$ in $\pi_1(X_\alpha) \oplus \pi_1(X_\beta)$ such that $\pi_1(X_\alpha)/H_\alpha \oplus \pi_1(X_\beta)/H_\beta$ is solvable.

Next, there is a solvable $\pi_1(X_\alpha)/H_\alpha$- action on $\pi_n(X_\alpha)$ for $n \geq 2$, and a solvable $\pi_1(X_\beta)/H_\beta$- action on $\pi_n(X_\beta)$ for $n \geq 2$ respectively.

Now we get the componentwise solvable action $\pi_1(X_\alpha)/H_\alpha \oplus \pi_1(X_\beta)/H_\beta$ on $\pi_n(X_\alpha) \oplus \pi_n(X_\beta)$. Thus the finite product space $\prod_{\alpha \in M} X_\alpha \in T_{RS}$.

Conversely, if $X_\alpha \times X_\beta$ is a residually sovable space, for any element $(g_\alpha, g_\beta)(\neq 1) \in \pi_1(X_\alpha) \oplus \pi_1(X_\beta)$ there are nontrivial normal subgroups $H_\alpha \oplus H_\beta$ in $\pi_1(X_\alpha) \oplus \pi_1(X_\beta)$ such that $\pi_1(X_\alpha)/H_\alpha \oplus \pi_1(X_\beta)/H_\beta$ is solvable. Thus we have nontrivial normal subgroups $(g_\alpha \notin)H_\alpha$ and $(g_\beta \notin)H_\beta$ in $\pi_1(X_\alpha)$ and $\pi_1(X_\beta)$ respectively such that $\pi_1(X_\alpha)/H_\alpha$ and $\pi_1(X_\beta)/H_\beta$ are solvable. Thus $\pi_1(X_\alpha)$ and $\pi_1(X_\beta)$ are solvable.

Next, from the componentwise $\pi_1(X_\alpha) \oplus \pi_1(X_\beta)$-action on $\pi_n(X_\alpha) \oplus \pi_n(X_\beta)$, and by the projection onto the each factor, we get the solvable $\pi_1(X_\alpha)$-action on $\pi_n(X_\alpha)$. Thus we have X_α as a residually nilpotent space for any α.

And T_S is also a full subcategory of T_{RS} naturally.

We know the following [9]: for the ordered medium V, there are no topologically stable

point defects if $\pi_2(V) = 0 \cdots (1)$

line defects if $\pi_1(V) = 0 \cdots (2)$

wall defects if $\pi_0(V) = 0 \cdots (3)$

For the proof of (2) above we may replace the possible line defect Σ_1 by a cylinder region C of cross-section D. The order parameter ϕ is now well defined on the boundary of D with Σ_1 inside C. For a closed loop γ on the boundary of C we have $\phi(\gamma) : S^1 \to V$ and so we get $[\phi(\gamma)] \in \pi_1(V)$. If $\pi_1(V) = 0$ then $[\phi(\gamma)] = [c]$ where c is a constant map. Thus $\phi(\gamma)$ is homotopic to the constant map and the order parameter function can extended over D, where D is the unit ball in R^2, which means that a topologically stable line defect is impossible.

We can prove (1) above by the similar procedure as in the case of (2), namely replacing a possible point defect Σ at P by a spherical region containing Σ at P. For the order parameter map $\phi : S^2 \to V$ then we get $[\phi] \in \pi_1(V)$. If $\pi_2(V) = 0$ then ϕ is homotopic to a constant map. Thus there exists an order parameter function ϕ which can be extended over D^2, where D^2 is the unit ball in R^3, which means that a topologically stable point defect is impossible.

Finally for the case (3), from the fact that $\pi_0(V) = 0$ we know the space V is path-connected. Thus a path from a point p, say on one side of the wall to a point q, say on the other side along which ϕ is well defined, must exist. This path effectively represents a hole in the wall which can then be made to disappear by continuity. Thus when $\pi_0(V) = 0$, wall defects are topologically unstable.

3. Main Theorems

Let's focus on the order parameter space X as a kind of topological space with respect to the defects and textures. We know the Klein bottle is an order parameter space [9].

THEOREM 3.1. *Let K be the Klein bottle. If $f : X \to K$ is a quasi-nilpotent homology equivalence and X is one of the followings;*

(1) $X(\in T_{RS})$,
(2) *X is the space satisfying condition (T^*) or (T^{**}) with $\pi_1(X)$ finite,*

then X also has the same state with the Klein bottle with respect to defects.

Proof. By the classical homotopy exact sequence of fibration: $F_f \to X \xrightarrow{f} K$, $\pi_1(f)$ is an epimorphism. And from the quasi-nilpotent homology equivalence of f we get $\tilde{H}_1(F_f)$ is trivial, i.e., $\pi_1(F_f)$ is a perfect group. Thus $\pi_1(K) \cong \pi_1(X)/P\pi_1(X)$ where $P\pi_1(X)$ means a perfect normal subgroup of $\pi_1(X)$. Now let's check the each case.

For the case (1): since $X \in T_{RS}$, by the definition of the residual solvability of the space X we have

$P(\pi_1(X)/N) = (P(\pi_1(X)/N))^{(n)} \leq (\pi_1(X)/N)^{(n)}$

and the last term must be trivial and finally $P(\pi_1(X)/N)$ is trivial. Thus $P\pi_1(X) = 0$.

For the case (2): from the fact that X satisfies the conditon (T^{**}) with $\pi_1(X)$ finite we get the space $\pi_1(X)$ is trivial [4, Lemma 3.1].

Next, if X satisfies the condition (T^*) then we know that X also satisfies the condition (T^{**}) [5,6]. Thus our proof is completed via a homotopy equivalence of f and the Whitehead theorem.

COROLLARY 3.2. *If $f : X \to K$ is an acyclic map and the space X is one of the cases* (1) $\sim$ (2) *of Theorem 3.1, then X also has the same state with the Klein bottle with respect to the defects.*

THEOREM 3.3. *If $f : X \to Y$ is an acyclic map, where X and Y are 3-dimensional ordered parameters V_1 and V_2 respectively. Furthermore X satisfies one of the conditions* (1) $\sim$ (2) *of Theorem 3.1 then X and Y have the same defects.*

Proof. By the Theorem 3.1 and the property of the homotopy equivalence, our proof is completed.

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