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THE WEAK LAW OF LARGE NUMBER FOR NORMED WEIGHTED SUMS OF STOCHASTICALLY DOMINATED AND PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES

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Abstract Let $\{X_n, n \geq 1\}$ be a sequence of pairwise negative quadrant dependent (NQD) random variables which are stochastically dominated by X. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $a_n > 0$ and $0 < b_n \to \infty$. In this note a weak law of large numbers of the form $(\sum_{j=1}^{n} a_j X_j - \nu_n)/b_n \xrightarrow{P} 0$ is established, where $\{\nu_n, n \geq 1\}$ is a suitable sequence.

1. Introduction

A sequence $\{X_n, n \geq 1\}$ of random variables is called positively(negatively) quadrant dependent (PQD(NQD)) if for each pair $i, j (i \neq j)$ and for all $r_i, r_j \in \mathbf{R}\, P\{X_i > r_i, X_j > r_j\} \geq (\leq) P\{X_i > r_i\} P\{X_j > r_j\} (or P\{X_i \leq r_i, X_j \leq r_j\} \geq (\leq) P\{X_i \leq r_i\} P\{X_j \leq r_j\})$ holds. This definition was introduced by Lehmann (1966).

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{a_n, n \geq 1\}$, $\{b_n, n \geq 1\}$ sequences of constants with $a_n \neq 0, 0 < b_n \to \infty$. Let $\{\nu_n, n \geq 1\}$ be a suitable sequence. Then $\{a_n X_n, n \geq 1\}$ is

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said to obey the general weak law of large numbers(WLLN) with centering constants (or random variables) $\{\nu_n, n \geq 1\}$ and norming constants $\{b_n, n \geq 1\}$ if the normed and centered weighted sum $(\sum_{j=1}^{n} a_j X_j - \nu_n)/b_n$ converges in probability to 0.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant $D < \infty$ such that

$$P(|X_n| > t) \leq DP(|DX| > t) \text{ for } t \geq 0 \text{ and } n \geq 1. \tag{1}$$

Chow and Teicher(1988) proved the classical WLLN for independent, identically distributed random variables $\{X_n,\ n \geq 1\}$ attributed to Feller(1946) and Adler, Rosalsky and Taylor(1991) generalized an earlier WLLN of Adler and Rosalsky(1991) which dealt with iid random variables, that is, Adler, Rosalsky and Taylor(1991) derived conditions on the behaviors of $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ and on the distribution of X ensuring that $\{a_n X_n, n \geq 1\}$ obeys the WLLN for the independent random varables which are stochastically dominated by X.

In this note, we investigate the WLLN for pairwise NQD random variables which are stochastically dominated bya random variable X using the tools which Adler, Rosalsky and Taylor(1991) have used to prove the WLLN for the a sequence of stochastically dominated independent random variables.

2. Preliminaries

LEMMA 2.1 **(Adler and Rosalsky, 1987)**. *Let X_0 and X be random variables such that X_0 is stochastically dominated by X in the sense that for some positive constant $0 < D < \infty$*

$$P\{|X_0| > t\} \leq DP\{|DX| > t\},\ t \geq 0.$$

Then for all $p > 0$ and $t \geq 0$

$$E|X_0|^p I(|X_0| \leq t) \leq Dt^p P\{|DX| > t\} + D^{p+1} E|X|^p I(|DX| \leq t). \tag{2}$$

LEMMA 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X in the sense that (1) holds. Let $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ be sequences of constants with $a_n > 0,\ 0 < b_n \to \infty$. Put $c_n = b_n/a_n$ If*

$$nP(|DX| > c_n) = o(1), \tag{3}$$

then

$$\frac{\sum_{j=1}^{n} a_j(X_j - X_{nj})}{b_n} \xrightarrow{P} 0 \tag{4}$$

obtains, where

$$\begin{aligned} X_{nj} &= X_j I[|X_j| \leq c_n] + c_n I[X_j > c_n] \\ &\quad - c_n I[X_j < -c_n],\ i \leq j \leq n,\ \text{for } c_n \geq 0. \end{aligned} \tag{5}$$

Proof. For arbitray $\epsilon > 0$,

$$\begin{aligned} P\left\{\frac{|\sum_{j=1}^{n} a_j(X_j - X_{nj})|}{b_n} > \epsilon\right\} &\leq P\{\cup_{j=1}^{n}[X_j \neq X_{nj}]\} \\ &\leq \sum_{j=1}^{n} P\{|X_j| > c_n\} \\ &= DnP\{|DX| > c_n\} = o(1) \end{aligned}$$

by (2). Hence the desired result follows.

REMARK. Note that if $\{X_n,\ n \geq 1\}$ is a sequence of pairwise NQD random variables then $\{X_{nj},\ 1 \leq j \leq n,\ n \geq 1\}$ defined in (5) is also a sequence of pairwise NQD by Lemma 2 of Matula(1992).

By modifying methods of proof of Theorem of Adler, Rosalsky and Taylor(1991) we obtain the following lemma without independence assumption:

LEMMA 2.3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X in the sense that (1) holds. Let $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ be* *sequences of constants with* $a_n > 0,\ 0 < b_n \to \infty,\ n \geq 1$, *and let* $p = 1, 2$. *Suppose that either*

$$\frac{b_n}{a_n} \uparrow, \quad \frac{b_n}{na_n} \downarrow, \quad \sum_{j=1}^{n} a_j^p = o(b_n^p),$$

and

$$\sum_{j=1}^{n} \frac{b_j^p}{j^2 a_j^p} = O\left(\frac{b_n^p}{\sum_{j=1}^{n} a_j^p}\right) \tag{6}$$

or

$$\frac{b_n}{a_n} \uparrow, \quad \frac{b_n}{na_n} \to \infty,$$

$$\sum_{j=1}^{n} a_j^p = O(na_n^p), \text{ and } \sum_{j=1}^{n} \frac{b_j^p}{j^p a_j^p} = O\left(\frac{b_n^p}{\sum_{j=1}^{n} a_j^p}\right) \tag{7}$$

hold. Then (3) entails that

$$\sum_{j=1}^{n} a_j^p P\{|DX| > c_n\} = o(a_n^p) \tag{8}$$

and

$$\sum_{j=1}^{n} a_j^p E|X|^p I(|DX| \leq c_n) = o(b_n^p) \tag{9}$$

hold, where $c_n = b_n/a_n$.

3. Main Results

THEOREM 3.1. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of pairwise NQD random variables which are stochastically dominated by a random variable* X *in the sense that (1) holds. Let* $\{a_n,\ n \geq 1\}$ *and* $\{b_n,\ n \geq 1\}$ *be constants with* $a_n > 0 < b_n \to \infty$, *and suppose that either (6) or (7) hold for* $p = 2$. *If (3) holds and* X_{nj} *is defined as in Lemma 2.2 then the WLLN*

$$\frac{\sum_{j=1}^{n} a_j (X_{nj} - EX_{nj})}{b_n} \xrightarrow{P} 0$$

obtains.

Proof. First note that $\{X_{nj}\}$ is a sequence of pairwise NQD by Lemma 2 in Matula(1992). It is enough to show that for arbitrary $\epsilon > 0$,

$$P\left\{\frac{|\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj})|}{b_n} > \epsilon\right\} \to 0 \text{ as } n \to \infty. \tag{10}$$

Since X_{nj}'s are pairwise NQD we have

$$\begin{aligned}
&\frac{1}{b_n^2} E\left|\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj})\right|^2 \\
&\le \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 E(X_{nj} - EX_{nj})^2 \text{ (by pairwise NQD)} \\
&\le \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 E(X_{nj}^2) \\
&\le \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 EX_j^2 I(|X_j| \le c_n) \\
&\qquad + \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 c_n^2 P\{|X_j| > c_n\} \text{ by (5)} \qquad (11) \\
&\le \frac{2}{b_n^2}\sum_{j=1}^{n} a_j^2[(Dc_n^2 P\{|DX| > c_n\}) \\
&\qquad + D^3 E|X|^2 I(|DX| \le c_n)] \text{ by (1) and (2)} \\
&= \frac{2D}{a_n^2}\sum_{j=1}^{n} a_j^2 P\{|DX| > c_n\} \\
&\qquad + \frac{2D^3}{b_n^2}\sum_{j=1}^{n} a_j^2 E|X|^2 I(|DX| \le c_n) \\
&= o(1) \qquad \text{by (8) and (9) for } p = 2.
\end{aligned}$$

Thus from Chebyshev's inequlity the desired result (10) follows.

From Lemma 2.2 and Theorem 3.1 we obtain a weak law of large numbers of the form $(\sum_{j=1}^{n} a_j X_j - \nu_n)/b_n \xrightarrow{P} 0$ is established, where $\{\nu_n, n \geq 1\}$is a suitable sequence.

THEOREM 3.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of pairwise NQD random variables which are stochastically dominated by a random variable X in the sense that (1) holds. Let $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ be constants with $a_n > 0,\ 0 < b_n \to \infty,\ n \geq 1$, and suppose that either for (6) or (7) holds for $p = 2$. If (3) holds then the WLLN*

$$\frac{\sum_{j=1}^{n} a_j (X_j - EX_{nj})}{b_n} \xrightarrow{P} 0 \tag{12}$$

obtains, where $X_{nj} = X_j I(|X_j| \leq c_n) + c_n I(X_j > c_n) - c_n I(X_j < -c_n),\ 1 \leq j \leq n,\ n \geq 1$ and $c_n = b_n/a_n$.

THEOREM 3.3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X in the sense that (1) holds. Let $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ be sequences of constants with $a_n > 0,\ 0 < b_n \to \infty$, and suppose that either (6) or (7) hold for $p = 1$. Then (3) entails that*

$$\sum_{j=1}^{n} a_j EX_{nj} \ / \ b_n \xrightarrow{P} 0 \tag{13}$$

holds.

Proof.

$$
\begin{aligned}
\frac{1}{b_n}\left|\sum_{j=1}^{n} a_j E X_{nj}\right| &\le \frac{1}{b_n}\sum_{j=1}^{n} a_j E|X_{nj}| \\
&= \frac{1}{b_n}\sum_{j=1}^{n} a_j E|X_j| I(|X_j| \le c_n) \\
&\quad + \frac{c_n}{b_n}\sum_{j=1}^{n} a_j P\{|X_j| > c_n\} \\
&\le \frac{D}{a_n}\sum_{j=1}^{n} a_j P\{|DX| > c_n\} \\
&\quad + \frac{D^2}{b_n}\sum_{j=1}^{n} a_j E|X| I(|DX| \le c_n) \\
&= o(1) \qquad \text{by (7) and (8) for } \ p = 1.
\end{aligned}
$$

Thus the desired result follows.

Finally, from Theorems 3.2 and 3.3 Remark is obtained.

REMARK. Let $\{X_n,\ n \ge 1\}$ $\{a_n,\ n \ge 1\}$ and $\{b_n,\ n \ge 1\}$ be defined as in Theorem 3.3 and suppose that either (6) or (7) holds for $p = 1$ and $p = 2$. Then (3) entails that

$$\sum_{j=1}^{n} a_j X_j \ / \ b_n \xrightarrow{P} 0$$

holds.

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