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A NOTE ON GRÖBNER FANS OF IDEALS

YONGGU KIM

Dept. of Mathematics Education,
Chonnam National University, Kwangju 500-757, Korea.
E-mail : kimm@chonnam.chonnam.ac.kr.

Abstract In this paper we reproduce results on Gröbner fans of ideals following a paper by T. Mora and L. Robbiano [5], and introduce stable Gröbner fans of ideals introduced by D. Mall [4]. We make minor corrections for the clarification, simplify proofs and provide new proofs. At the end we give a description of Gröbner fans of toric ideals.

1. Introduction

Let V be a vector space over k, a field of characteristic 0, with a basis $x_1, \ldots, x_n$, and let $S = \bigoplus_k S^k V$, the symmetric algebra on V, that is, $S = k[x_1, \ldots, x_n]$. Ideals of polynomial ring $S = k[x_1, \ldots, x_n]$ have finite generating bases. Especially Gröbner bases (see Definition 3.2) give simple and effective method for algorithmic solution of various problems in algebraic geometry and commutative algebra. Gröbner bases may be considered as finite representations of polynomial ideals. However they are not invariants of the ideals but depend on the chosen term orderings (Definition 2.8). Every ideal has only finitely many distinct initial ideals (Proposition 3.6), and thus has finitely many distinct reduced Gröbner bases. From a finite set of reduced Gröbner bases, we obtain an invariant of a polynomial ideal; a Gröbner fan (Definition 3.8). A Gröbner fan of an ideal describes term orderings in terms of corresponding initial ideals. In this paper we intend to introduce and describe Gröbner fans and stable Gröbner fans of ideals.

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The organization of the paper is as follows. In section 2 we give a description of classification of orders following a paper by T. Mora and L. Robbiano [5], simplifying proofs and giving details just for our needs. In section 3 we first describe Gröbner fans of ideals, and then stable Gröbner fans of ideals introduced by D. Mall [4] with minor corrections for the clarification. One of main purposes is to make clear how term orders are related in a Gröbner fan. We also raise a few questions which look interested for further works on stable Gröbner fans. At the last section toric ideals are introduced and the description of stable Gröbner fans of a special class of toric ideals are given (Proposition 4.8). At last I would like to express my sincere thanks to D. Mall for his kind explanation for my questions on his paper.

2. Term Orderings

For a vector $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$, we define $V_{\mathbf{u}}$ to be the $\mathbb{Q}$-subvector space of $\mathbb{R}$ spanned by u_i, that is $V_{\mathbf{u}} = \text{Span}_{\mathbb{Q}}(u_1, \cdots, u_n)$. Let $d(\mathbf{u}) := \dim_{\mathbb{Q}} V_{\mathbf{u}}$, the dimension of $V_{\mathbf{u}}$. From now on we assume that V is a $\mathbb{Q}$-subvector space of $\mathbb{Q}^n$ of dimension d. Let $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ an array of vectors in $V_{\mathbb{R}}$, where $\mathbf{u}_i = (u_{i1}, \cdots, u_{in}) \in V_{\mathbb{R}} \subset \mathbb{R}^n$. Let $V_{\mathbf{u}_i} := \text{Span}_{\mathbb{Q}}(u_{i1}, \cdots, u_{in})$ and $d_i = \dim_{\mathbb{Q}} V_{\mathbf{u}_i}$.

For each $\mathbf{u}_i$ of an array of vectors $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$, we intend to associate a uniquely determined set of rational vectors $\{\mathbf{u}_{i1}, \cdots, \mathbf{u}_{id_i}\}$ in V, $d_i = \dim_{\mathbb{Q}} V_{\mathbf{u}_i}$. Let $\{\lambda_{i1}, \cdots, \lambda_{id_i}\}$ be a $\mathbb{Q}$-basis of $V_{\mathbf{u}_i}$, $i = 1, \cdots, s$. Since $u_{ij} \in V_{\mathbf{u}_i}$, u_{ij} has a unique representation as a linear combination of λ_{il} over $\mathbb{Q}$. Thus $u_{ij} = \sum_{l=1}^{d_i} q_{ij\,l}\lambda_{il}$, where $q_{ij\,l} \in \mathbb{Q}$ are uniquely determined rational numbers. Then

$$\begin{aligned}
\mathbf{u}_i &= (u_{i1}, \cdots, u_{in}) \\
&= (\sum_{l=1}^{d_i} q_{i1l}\,\lambda_{il}, \cdots, \sum_{l=1}^{d_i} q_{inl}\,\lambda_{il}) \\
&= \sum_{l=1}^{d_i} \lambda_{il}\,(q_{i1l}, \cdots, q_{inl}) = \sum_{l=1}^{d_i} \lambda_{il}\,\mathbf{u}_{il},
\end{aligned}$$

where $\mathbf{u}_{il} = (q_{i1l}, \cdots, q_{inl}) \in V \subset \mathbb{Q}^n$, $l = 1, \cdots, d_i$ are uniquely determined vectors depending on the basis $\{\lambda_{i1}, \cdots, \lambda_{id_i}\}$.

DEFINITION 2.1. As before, let $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ be an array of vectors in $V_\mathbb{R}$ for a d-dimensional $\mathbb{Q}$-subvector space V of $\mathbb{Q}^n$. We observed that for each $\mathbf{u}_i$ there corresponds a set of rational vectors $\{\mathbf{u}_{i1}, \cdots, \mathbf{u}_{id_i}\}$ in V. Let

$$\mathcal{U}_\mathbb{Q} = (\mathbf{u}_{11}, \cdots, \mathbf{u}_{1d_1}, \mathbf{u}_{21}, \cdots, \mathbf{u}_{2d_2}, \cdots, \mathbf{u}_{s1}, \cdots, \mathbf{u}_{sd_s}).$$

(1) The array $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ is rationally independent if the $\sum d_i$-tuple $\mathcal{U}_\mathbb{Q}$ is an array of linearly independent vectors of V over $\mathbb{Q}$.
(2) The set $\{\mathbf{u}_1, \cdots, \mathbf{u}_s\}$ is a set of rational generators of V if $\mathcal{U}_\mathbb{Q}$ is a set of generators of V over $\mathbb{Q}$.
(3) The array $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ is a rational basis of V if the $\sum d_i$-tuple $\mathcal{U}_\mathbb{Q}$ is a basis of V over $\mathbb{Q}$.

PROPOSITION 2.2. *Let $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$, where $\mathbf{u}_i \in V_\mathbb{R}$ and V is a d-dimensional $\mathbb{Q}$-subvector space of $\mathbb{Q}^n$. Then*

(i) *$\{\mathbf{u}_1, \cdots, \mathbf{u}_s\}$ is a set of rational generators of V if and only if*

$$(*) \qquad \mathbf{u} \in V \text{ and } \mathbf{u} \cdot \mathbf{u}_1 = \cdots = \mathbf{u} \cdot \mathbf{u}_s = 0 \Longrightarrow \mathbf{u} = 0.$$

(ii) *$(\mathbf{u}_1, \cdots, \mathbf{u}_s)$ is a rational basis of V if and only if $(*)$ holds and $\sum d(\mathbf{u}_i) = d$.*

Proof. (i) We first note that for a vector $\mathbf{u} \in V$, the followings are true:
(**)

$$\mathbf{u} \cdot \mathbf{u}_i = 0 \Longleftrightarrow \sum \lambda_{ij} \mathbf{u} \cdot \mathbf{u}_{ij} = 0 \Longleftrightarrow \mathbf{u} \cdot \mathbf{u}_{ij} = 0,\ j = 1, \cdots, d_i.$$

The third relation follows from the fact that $\{\lambda_{i1}, \cdots, \lambda_{id_i}\}$ is a basis of $V_{\mathbf{u}_i}$ over $\mathbb{Q}$ for each $i = 1, \cdots, s$ and $\mathbf{u} \cdot \mathbf{u}_{ij} \in \mathbb{Q}$.
($\Rightarrow$) Suppose that $\{\mathbf{u}_1, \cdots, \mathbf{u}_s\}$ is a set of rational generators of V and $\mathbf{u} \cdot \mathbf{u}_i = 0$, $1 \le i \le s$, for some $\mathbf{u} \in V$. Then by (**) we conclude that $\mathbf{u} \cdot \mathbf{u}_{ij} = 0$, $\forall\, i, j$. Since $\{\mathbf{u}_{ij}\}$ is a set of generators of V, it follows that $\mathbf{u} = 0$.

($\Leftarrow$) For every $\mathbf{u} \in V$, let $\mathbf{v} = \mathbf{u} - \sum(\mathbf{u}\cdot\mathbf{u}_{ij})/(\mathbf{u}_{ij}\cdot\mathbf{u}_{ij})\ \mathbf{u}_{ij}$. We note that $\mathbf{v}\cdot\mathbf{u}_{ij} = 0$ for all i, j. Then (**) implies that $\mathbf{v}\cdot\mathbf{u}_i = 0$ for all i, and by the hypothesis $\mathbf{v} = 0$. Thus $\mathbf{u} = \sum(\mathbf{u}\cdot\mathbf{u}_{ij})/(\mathbf{u}_{ij}\cdot\mathbf{u}_{ij})\ \mathbf{u}_{ij}$, which implies that $\{\mathbf{u}_{ij}\}$ is a set of generators of V over $\mathbb{Q}$.
(ii) This follows from (i).

We now describe a procedure which produces an array of rationally independent vectors from a given array of vectors. Let $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ be an array of vectors. Then as before we get an array of rational vectors $\mathcal{U}_{\mathbb{Q}} = (\mathbf{u}_{11}, \cdots, \mathbf{u}_{sd_s})$. If $\{\mathbf{u}_{11}, \cdots, \mathbf{u}_{sd_s}\}$ is a set of linearly independent vectors, then we are done. Suppose that it is not the case and $\mathbf{u}_{11} \neq 0$, then from the elementary linear algebra some $\mathbf{u}_{ij}$ $(i > 1, j > 1)$ is a linear combination of the preceding vectors, $\mathbf{u}_{11}, \cdots, \mathbf{u}_{i(j-1)}$. We eliminate the vector $\mathbf{u}_{ij}$ from the array $\mathcal{U}$, and check if $\{\mathbf{u}_{11}, \cdots, \widehat{\mathbf{u}_{ij}}, \cdots, \mathbf{u}_{sd_s}\}$ is linearly dependent, where $\widehat{\mathbf{u}_{ij}}$ means that the vector $\mathbf{u}_{ij}$ is eliminated from the list. By doing this elimination process inductively, we get an array of rational vectors $(\mathbf{u}_{11}, \cdots, \mathbf{u}_{te_t})$, where $t \leq s$ and $e_t \leq d_t$. We eliminate $\mathbf{u}_i$ from the array of vectors $\mathcal{U} = (\mathbf{u}_i, \cdots, \mathbf{u}_s)$ if all vectors $\mathbf{u}_{ij}$, for $j = 1, \cdots, d_i$ are eliminated.

An array of vectors $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ is called essential if the above procedure does not eliminate any vector of $\mathcal{U}$. We note that an essential array of vectors is not necessary an array of rationally independent vectors. The condition being essential only guarantees that the length of an array of vectors does not change after the above elimination process. From now on we assume that all array of vectors are essential.

DEFINITION 2.3. An order $\prec$ on V is a total order which is compatible with the additive group structure of V, that is, if $\mathbf{u} \prec \mathbf{v} \Longrightarrow \mathbf{u} + \mathbf{w} \prec \mathbf{v} + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $\{\mathbf{u} \in V \mid 0 \prec \mathbf{u}\}$ spans V. We denote $\mathcal{TO}(V)$ to be the set of orders on V.

Let $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ be an array of vectors of $V_{\mathbb{R}}$. Then we define a $\mathcal{U}$-degree map to be the map $\deg_{\mathcal{U}} : V \longrightarrow \mathbb{R}^s$, where

$$\deg_{\mathcal{U}}(\mathbf{v}) = (\mathbf{v}\cdot\mathbf{u}_1, \cdots, \mathbf{v}\cdot\mathbf{u}_s) \in \mathbb{R}^s,\ \mathbf{v} \in V.$$

Next we assign lexicographic order or simply **lex** order to $\mathbb{R}^s$ as follows: $0 = (0, \cdots, 0) < (r_1, \cdots, r_s)$ if and only if the left-most nonzero entry r_i is positive. Then the $\mathcal{U}$-degree map $\deg_{\mathcal{U}}$ induces an order $\prec_{\mathcal{U}}$ on V, which is called an order of linear type, as follows:

$$\forall \mathbf{v} \in V,\ 0 \prec_{\mathcal{U}} \mathbf{v} \Longleftrightarrow 0 < \deg_{\mathcal{U}}(\mathbf{v})$$

By Proposition 2.2, we note that the induced order $\prec_{\mathcal{U}}$ with $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ is a total order if and only if the set $\{\mathbf{u}_1, \cdots, \mathbf{u}_s\}$ is a set of rational generators of V. Next we intend to show that every order on V is induced from an array of vectors whose the underlying set is a set of rational generators of V (Proposition 2.6).

LEMMA 2.4. *Let V be a $\mathbb{Q}$-subvector space of $\mathbb{Q}^n$ of dimension d and $\prec$ an order on V. Set $V^+ := \{\mathbf{v} \in V \mid 0 \prec \mathbf{v}\}$, $V^- := \{\mathbf{v} \in V \mid \mathbf{v} \prec 0\}$ and $I_V := \overline{(V^+)_{\mathbb{R}}} \cap \overline{(V^-)_{\mathbb{R}}}$, where $\overline{X}$ means a closure of the set X in the usual Euclidean topology on $V_{\mathbb{R}}$. Then*

(i) *I_V is a $(d-1)$ dimensional subspace of $V_{\mathbb{R}}$.*
(ii) *For $\mathbf{u} \in V_{\mathbb{R}}$, $d(\mathbf{u}) \leq d = dim_{\mathbb{Q}} V$.*

Proof. (i) Let $\mathbf{u}, \mathbf{v} \in I_V$. Then to show that I_V is a subspace of $V_{\mathbb{R}}$, we must show that $\mathbf{u} + \mathbf{v} \in I_V$ and $k\,\mathbf{u} \in I_V$, $k \in \mathbb{R}$. $\mathbf{u} + \mathbf{v} \in I_V$ if and only if for all $\epsilon > 0$, $B(\mathbf{u} + \mathbf{v}, \epsilon) \bigcap V^+ \neq \emptyset$ and $B(\mathbf{u} + \mathbf{v}, \epsilon) \bigcap V^- \neq \emptyset$, where $B(\mathbf{a}, \epsilon)$ is an open ball with a center $\mathbf{a}$ and a radius ϵ. Since $\mathbf{u}, \mathbf{v} \in I_V$, there exit $\mathbf{u}_1, \mathbf{v}_1 \in V$ such that $\mathbf{u}_1 \in B(\mathbf{u}, \epsilon/2) \bigcap V^+ \neq \emptyset$ and $\mathbf{v}_1 \in B(\mathbf{v}, \epsilon/2) \bigcap V^+ \neq \emptyset$. It is easy to check that $\mathbf{u}_1 + \mathbf{v}_1 \in V^+$ and $\|(\mathbf{u} + \mathbf{v}) - (\mathbf{u}_1 + \mathbf{v}_1)\| < \epsilon$. Then $\mathbf{u}_1 + \mathbf{v}_1 \in B(\mathbf{u} + \mathbf{v}, \epsilon) \bigcap V^+$, thus $B(\mathbf{u} + \mathbf{v}, \epsilon) \bigcap V^+ \neq \emptyset$ for all $\epsilon > 0$. Similarly $B(\mathbf{u} + \mathbf{v}, \epsilon) \bigcap V^- \neq \emptyset$ for all $\epsilon > 0$. Thus $\mathbf{u} + \mathbf{v} \in I_V$. Similarly it is easy to show that $k\,\mathbf{u} \in I_V$, $k \in \mathbb{R}$. Hence we conclude that I_V is a subspace of $V_{\mathbb{R}}$.

It is obvious that $(V^+)_{\mathbb{R}} \bigcap (V^-)_{\mathbb{R}} = \emptyset$ and $(V^+)_{\mathbb{R}} \bigcup (V^-)_{\mathbb{R}} = \mathbb{R} \backslash \{0\}$. Hence $V_{\mathbb{R}} \backslash I_V$ is disconnected. Therefore $\dim(I_V) = d-1$.

(ii) Let $\mathcal{B} = \{\mathbf{v}_1, \cdots \mathbf{v}_d\}$ be a basis of $V \subseteq \mathbb{Q}^n$. Then $\mathcal{B}$ is also a basis of $V_{\mathbb{R}}$. Let $\mathbf{u} = \sum_{i=1}^{d} \lambda_i \mathbf{v}_i$, $\lambda_i \in \mathbb{R}$. Since $V_{\mathbf{u}}$, the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by the coordinates of $\mathbf{u}$, is contained in a $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by $\{\lambda_1, \cdots, \lambda_d\}$, $d(\mathbf{u}) = \dim_{\mathbb{Q}} V_{\mathbf{u}} \leq d = \dim_{\mathbb{Q}} V$.

Let $\prec$ be an order on V, $\dim_{\mathbb{Q}}(V) = d$ and I_V the set defined at Lemma 2.4. Since I_V is a $(d-1)$-dimensional subspace of

$V_{\mathbb{R}}$, $V_{\mathbb{R}} \setminus I_V$ consists of two disjoint components which are either positive or negative with respect to the given order $\prec$. Let $\mathcal{R}(V) = \{\mathbf{v} \in V_{\mathbb{R}} : \mathbf{v}$ is orthogonal to I_V and $\mathbf{v} \in (V^+)_{\mathbb{R}}\}$. Then $\mathcal{R}(V)$ is an 1-dimensional ray in $V_{\mathbb{R}}$.

LEMMA 2.5. *Let $\mathbf{u}_1 \in \mathcal{R}(V)$ be a vector with the rational dimension $d(\mathbf{u}_1) = d_1$. Let $V_1 = I_V \cap V \subseteq \mathbb{Q}^n$. Then $dim_{\mathbb{Q}} V_1 = d - d_1$.*

Proof. Let $\mathbf{u}_1 = (\lambda_1, \cdots, \lambda_n)$ and $V_{\mathbf{u}_1} = \{\sum_{i=1}^n \lambda_i q_i : q_i \in \mathbb{Q}\}$, a $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by the coordinates of $\mathbf{u}_1$. Define a $\mathbb{Q}$-linear transformation $T : V \subseteq \mathbb{Q}^n \longrightarrow V_{\mathbf{u}_1} \subseteq \mathbb{R}$, $T(\mathbf{q}) = T(q_1, \cdots, q_n) = \mathbf{u}_1 \cdot \mathbf{q} = \sum_{i=1}^n \lambda_i q_i$. Then $\ker(T) = I_V \cap V = V_1$. Since T is a surjective linear transformation onto $V_{\mathbf{u}_1}$ and $\dim_{\mathbb{Q}} V_{\mathbf{u}_1} = d(\mathbf{u}_1) = d_1$, $\dim_{\mathbb{Q}} V_1 = \dim_{\mathbb{Q}} \ker(T) = \dim_{\mathbb{Q}} V - \dim_{\mathbb{Q}} V_{\mathbf{u}_1} = d - d_1$.

PROPOSITION 2.6.

(i) *For every order $\prec$ on V, there exists an array of rational basis $\mathcal{U}$ such that $\prec = \prec_{\mathcal{U}}$, that is, $\mathbf{v}_1 \prec \mathbf{v}_2$ if and only if $deg_{\mathcal{U}}(\mathbf{v}_1) < deg_{\mathcal{U}}(\mathbf{v}_2)$.*

(ii) *$\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ and $\mathcal{V} = (\mathbf{v}_1, \cdots, \mathbf{v}_t)$ are arrays of vectors and $\prec_{\mathcal{U}} = \prec_{\mathcal{V}}$, then there exists $\lambda \in \mathbb{R}^+$ such that $\mathbf{u}_1 = \lambda \mathbf{v}_1$.*

Proof. (i) By Lemma 2.5, $\dim_{\mathbb{Q}} V_1 = \dim_{\mathbb{Q}}(I_V \cap V) = d - d_1$, where $d_1 = d(\mathbf{u}_1)$ and $\mathbf{u}_1 \in \mathcal{R}(V)$. We note that for each $\mathbf{v} \in V \setminus V_1$, $0 \prec \mathbf{v}$ if and only if $0 < \mathbf{v} \cdot \mathbf{u}_1$. Let $(V_1)_{\mathbb{R}} = V_1 \otimes_{\mathbb{Q}} \mathbb{R}$ and $I_{V_1} = \overline{(V_1^+)_{\mathbb{R}}} \cap \overline{(V_1^-)_{\mathbb{R}}}$. By Lemma 2.4 $\dim I_{V_1} = d - d_1$. Let $\mathbf{u}_2 \in \mathcal{R}(V_1)$ which is orthogonal to $\mathbf{u}_1$. Then by Lemma 2.4, $d(\mathbf{u}_2) = d_2 \leq d - d_1 = \dim(V_1)$. Let $V_2 := I_{V_1} \cap V_1$. Then for each $\mathbf{v} \in V_1 \setminus V_2$, $\mathbf{v} \cdot \mathbf{u}_1 = 0$, and $0 \prec \mathbf{v}$ if and only if $0 < \mathbf{v} \cdot \mathbf{u}_2$. We repeat this process until we get a sequence of orthogonal vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_s\}$ and $d = \sum_{i=1}^s d(\mathbf{u}_i)$. From the above construction, $V = (V \setminus V_1) \uplus (V_1 \setminus V_2) \uplus \cdots \uplus (V_{(s-1)} \setminus V_s) \uplus V_s$, a union of disjoint sets, and thus it follows that $\prec = \prec_{\mathcal{U}}$.

(ii) Suppose that there does not exist any $\lambda \in \mathbb{R}^+$ such that $\mathbf{u}_1 = \lambda \mathbf{v}_1$. This means that $\mathbf{u}_1$ and $\mathbf{v}_1$ are not on the same ray in $\mathbb{R}^n$. Then we can choose a rational vector $\mathbf{u} \in \mathbb{Q}^n$ so that

$\mathbf{u} \cdot \mathbf{u}_1 \geq 0$ and $\mathbf{u} \cdot \mathbf{v}_1 < 0$. Thus $\prec_{\mathcal{U}} \neq \prec_{\mathcal{V}}$, which is contrary to the assumption.

Let $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ be an array of vectors of $V_{\mathbb{R}}$. Let W_i be the $\mathbb{Q}$-subspace of V consisted of vectors orthogonal to $(\mathbf{u}_1, \cdots, \mathbf{u}_i)$. Denote $(W_i)_{\mathbb{R}}$ to be $W_i \otimes_{\mathbb{Q}} \mathbb{R}$, and let $\mathcal{P}_i$ be the orthogonal projection from $V_{\mathbb{R}}$ to $(W_{(i-1)})_{\mathbb{R}}$, $i = 1, \ldots, s$.

THEOREM 2.7. *Let $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ and $\mathcal{V} = (\mathbf{v}_1, \cdots, \mathbf{v}_t)$ are arrays of vectors of $V_{\mathbb{R}}$. Then the followings are equivalent:*

(i) $\prec_{\mathcal{U}} = \prec_{\mathcal{V}}$

(ii) *$s = t$ and there exist positive real numbers λ_i such that $\mathcal{P}_i(\mathbf{u}_i) = \lambda_i \mathcal{P}_i(\mathbf{v}_i)$ for $i = 1, \ldots, s$.*

Proof. *(ii)* $\Longrightarrow$ *(i)* This follows from the definition of the order $\prec_{\mathcal{U}}$.

(i) $\Longrightarrow$ *(ii)* The case for $i = 1$ is obvious. We use the induction on i, and assume that $\mathcal{P}_k(\mathbf{u}_k) = \lambda_k \mathcal{P}_k(\mathbf{v}_k)$ for $k = 1, \ldots, \ell - 1$. Hence $\prec_{\mathcal{U}}$ and $\prec_{\mathcal{V}}$ induce the same ordering on W_ℓ, which we denote by $\prec_\ell$. Since $\mathcal{U}$ and $\mathcal{V}$ are arrays of vectors, $\prec_\ell$ considered as the order induced by $\prec_{\mathcal{U}}$ is represented by the first vector $\mathcal{P}_\ell(\mathbf{u}_\ell)$, on the other hand as the order induced by $\prec_{\mathcal{V}}$ is represented by the first vector $\mathcal{P}_\ell(\mathbf{v}_\ell)$. By Proposition 2.6, $\mathcal{P}_\ell(\mathbf{u}_\ell) = \lambda_\ell \mathcal{P}_\ell(\mathbf{v}_\ell)$, and hence the implication follows.

Let $S := k[x_1, \ldots, x_n]$, also denoted by $k[\mathbf{x}]$ for a short notation, be a polynomial ring in n variables over a field k. Let $\mathcal{T}_S$ be the set of monomial terms $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ in $k[\mathbf{x}]$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$. There is a monoid isomorphism $\mathbf{log} : \mathcal{T}_S \longrightarrow \mathbb{N}^n$ defined by $\mathbf{log}(\mathbf{x}^{\mathbf{a}}) = \mathbf{a}$. With this identification, it suffices to consider an order on $\mathbb{N}^n$ instead of considering an order on $\mathcal{T}_S$.

DEFINITION 2.8. A term order is an order $\prec$ on $\mathbb{Q}^n$ which are positive over $\mathbb{N}^n$, that is $\mathbf{0} \prec \mathbf{u}$ for all $\mathbf{u} \in \mathbb{N}^n$. Denote $\mathcal{TO}$ to be the set of term orders on $\mathbb{Q}^n$.

REMARK 2.9. A term order is also defined to be a total order $\prec$ on $\mathbb{N}^n$ satisfying the following properties : (i) $\mathbf{a} \prec \mathbf{b}$ implies $\mathbf{a}+\mathbf{c} \prec \mathbf{b}+\mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{N}^n$, that is, the order $\prec$ is compatible with the additive group structure on $\mathbb{N}^n$, (ii) The zero vector $\mathbf{0}$ is the unique minimal element, which is equivalent to the statement that the order $\prec$ is a well ordering. Every term order $\prec$ on $\mathbb{N}^n$ extends uniquely to $\mathbb{Z}^n$ and $\mathbb{Q}^n$. Thus it is easy to check that two definitions for a term order are identical.

3. Gröbner Fans of Ideals

DEFINITION 3.1. The support of f, denoted by $\mathrm{Supp}(f)$, is a set of monomial terms of f ignoring its coefficients. The monomial term $in_{\prec}(f) = \max_{\prec}\{t \mid t \in \mathrm{Supp}(f)\}$ is called an initial term of f with respect to a term order $\prec \in \mathcal{TO}$. The initial ideal of I, denoted by $in_{\prec}(I)$, is an ideal $\langle in_{\prec}(f) \mid f \in I \rangle$ which is generated by the initial term of f, $f \in I$.

DEFINITION 3.2. A finite set of polynomials $\mathcal{G} = \{f_1, \cdots, f_k\}$ is called a *Göbner basis* of an ideal I if the initial ideal of I, $in_{\prec}(I)$, is generated by a finite set $\{in_{\prec}(f_i) \mid f_i \in \mathcal{G}\}$, that is, $in_{\prec}(I) = \langle in_{\prec}(f_1), \ldots, in_{\prec}(f_n) \rangle$.

REMARK 3.3. (i) If $\mathcal{G} = \{f_1, \cdots, f_k\}$ is a Göbner basis, then $\mathcal{G}$ generates I, that is, $I = \langle f_1, \cdots, f_k \rangle$. A Göbner basis $\mathcal{G} = \{f_1, \cdots, f_k\}$ is called reduced if for each $i, 1 \le i \le k$, $in_{\prec}(f_i)$ does not divide any monomial term of f_j, $j \neq i$.
(ii) The reduced Göbner basis for a given ideal I and a term order $\prec$ is unique provided that the coefficients of every initial term of its polynomial is 1 ([2], [7]).

NOTATION 3.4. Let $E \subseteq \mathbb{Q}^n$, $\mathcal{B} \subseteq S$, and $\mathbf{m}$ a monomial ideal of S.

(1) $\mathcal{TO}(+, E) := \{\prec \in \mathcal{TO} \mid \mathbf{0} \prec \mathbf{u}, \forall \mathbf{u} \in E\}$, the set of orders in $\mathcal{TO}$ which are positive on E.
(2) $\mathcal{R}_{\prec}(I)$, the unique reduced Gröbner basis of an ideal I with respect to an order $\prec$.
(3) $\partial_{\prec}(f) := \{\mathbf{log}(in_{\prec}(f)) - \mathbf{log}(t) \mid t \in \mathrm{Supp}(f),\ t \neq in_{\prec}(f)\}$
(4) $\partial_{\prec}(\mathcal{B}) = \bigcup_{f \in \mathcal{B}} \partial_{\prec}(f)$.
(5) $\mathcal{TO}(I, \mathbf{m}) := \{\prec \in \mathcal{TO} \mid in_{\prec}(I) = \mathbf{m}\}$.

THEOREM 3.5. *(T. Mora and L. Robbiano [5]) For a given ideal I of S and a monomial ideal* $\mathbf{m}$, *let* $\prec \in \mathcal{TO}(I, \mathbf{m})$, *then*

(i) *For every order* $\prec' \in \mathcal{TO}(I, \mathbf{m})$, $\mathcal{R}_{\prec'}(I) = \mathcal{R}_{\prec}(I)$ *and*
(ii) $\mathcal{TO}(I, \mathbf{m}) = \mathcal{TO}(+, \partial_{\prec}(\mathcal{R}_{\prec}(I)))$.

The statement (i) of the above Theorem 3.5 says that if the initial ideal of I with respect to a term order $\prec'$ is identical to the initial ideal of I with respect to the term order $\prec$, then the corresponding reduced Gröbner basis of I with respect to two term orders are identical. While the statement (ii) says that those term orders with respect to which the initial ideals of I are identical to a given monomial ideal $\mathbf{m}$ are exactly those term orders who are positive on the set of difference vectors $\partial_{\prec}(\mathcal{R}_{\prec}(I))$, which means that the corresponding initial terms of each element of the reduced Gröbner basis of I, $\mathcal{R}_{\prec}(I)$, with respect to those term orders are identical to those initial terms with respect to the term order $\prec$.

Let $\mathcal{M}(I) = \{\mathbf{m} \mid \mathbf{m} = in_{\prec}(I), \prec \in \mathcal{TO}\}$, the set of initial ideals of I with respect to all term orders.

PROOSITION 3.6. *(T. Mora and L. Robbiano [5], B. Sturmfels [7]) For every ideal I of S, $\mathcal{M}(I)$ is a finite set.*

For a subset $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{R}^n$, let $\mathcal{C}(E) = \{\sum_{i=1}^k r_i \mathbf{u}_i \mid r_i \in \mathbb{R}^+\}$, the polyhedral cone of E, and $\mathcal{C}(E)^* = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{u}_i \geq 0,\ i = 1, \ldots, k\}$, the polar cone of E. By Proposition 3.6, we may assume that $\mathcal{M}(I) = \{\mathbf{m}_1, \ldots, \mathbf{m}_n\}$. Then for each i, $1 \leq i \leq n$, choose a term order $\prec_i \in \mathcal{TO}(I, \mathbf{m}_i)$ and the reduced Gröbner basis of I, $\mathcal{R}_{\prec_i}(I)$. For the simplicity, we use the following notations: $\mathcal{D}_i = \partial_{\prec_i}(\mathcal{R}_{\prec_i}(I))$ and $\mathcal{D}_i^* = \mathcal{C}(\mathcal{D}_i)^*$.

THEOREM 3.7. *(T. Mora and L. Robbiano [5]) Let I be an ideal of $S = k[x_1, \ldots, x_n]$. Then*

(i) *dim* $(\mathcal{D}_i^* \bigcap (\mathbb{R}^n)^+) = n$ *for each* $i = 1, \ldots, n$.
(ii) $(\mathbb{R}^n)^+ = \bigcup_i (\mathcal{D}_i^* \bigcap (\mathbb{R}^n)^+)$.
(iii) *For each* i, j $(i \neq j)$, $\mathcal{D}_i^* \bigcap \mathcal{D}_j^*$ *is a part of a proper face of* $\mathcal{D}_i^*$ *and* $\mathcal{D}_j^*$
(iv) *The followings are equivalent:*
(a) $\prec \in \mathcal{TO}(+, \mathcal{D}_i)$

(b) $in_{\prec}(I) = \mathbf{m}_i$ *and* $\mathcal{R}_{\prec_i}(I)$ *is the reduced Gröbner basis of* I *with respect to* $\prec$.

Here we give a description of a Gröbner cone $\mathcal{D}_i^*$ in terms of term orders. Let $\mathbf{u} \in (\mathcal{D}_i^*)^\circ$, the interior of the cone $\mathcal{D}_i^*$, and consider the induced order $\prec_{\mathcal{U}}$ for an array of vectors $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$ with $\mathbf{u}_1 = \mathbf{u}$. Then the order $\prec_{\mathcal{U}}$ is positive on the difference vectors $\mathcal{D}_i$, that is, $\prec_{\mathcal{U}}(g) = \prec_i(g)$ for all $g \in \mathcal{R}_{\prec_i}(I)$. By Theorem 3.7, it follows that $in_{\prec_{\mathcal{U}}}(I) = \mathbf{m}_i$ and $\mathcal{R}_{\prec_{\mathcal{U}}}(I) = \mathcal{R}_{\prec_i}(I)$. If $\prec_{\mathcal{U}} = \prec_{\mathcal{V}}$ for another array of vectors $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$, then by Proposition 2.6 $\mathbf{v}_1 = \lambda\, \mathbf{u}_1$, $\lambda \in \mathbb{R}^+$.

Now conversely suppose that for an order $\prec$, $in_{\prec}(I) = \mathbf{m}_i$ and $\mathcal{R}_{\prec}(I) = \mathcal{R}_{\prec_i}(I)$. Then by Theorem 3.7, $\prec \in \mathcal{TO}(+, \mathcal{D}_i)$. By Proposition 2.6, there exists an array of vectors $\mathcal{U}$ such that $\prec = \prec_{\mathcal{U}}$, $\mathcal{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_s)$. From the fact that $\prec_{\mathcal{U}} = \prec \in \mathcal{TO}(+, \mathcal{D}_i)$, we conclude that $\mathbf{u}_1 \in (\mathcal{D}_i^*)^\circ$. Similarly for every array of vectors $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$ with $\prec = \prec_{\mathcal{V}}$, $\mathbf{v}_1 \in (\mathcal{D}_i^*)^\circ$.

Hence rays determined by vectors $\mathbf{u} \in (\mathcal{D}_i^*)^\circ$ represent precisely those orders whose initial ideal and the reduced Gröbner basis of I are $\mathbf{m}_i$ and $\mathcal{R}_{\prec_i}(I)$ respectively.

DEFINITION 3.8. Let $\mathcal{M}(I) = \{\mathbf{m}_1, \ldots, \mathbf{m}_n\}$. Then the restricted *Gröbner fan* of I is $\mathcal{F}^+(I) = \{\mathcal{D}_1^* \cap \mathbb{R}_+^{n+1}, \ldots, \mathcal{D}_n^* \cap \mathbb{R}_+^{n+1}\}$

We now assume that every ideal I is a homogeneous ideal of a polynomial ring $S = k[x_0, \ldots, x_n]$. Then it is well known that for every term order $\prec$ and a homogeneous ideal I of S, the reduced Gröbner basis of I, $\mathcal{R}_{\prec}(I)$ consists of homogeneous polynomials.

REMARK 3.9. For every vector $\mathbf{u} \in \partial_{\prec}(I)$, $\mathbf{u} \cdot (1, \ldots, 1) = 0$.

DEFINITION 3.10. The *Gröbner fan* of an ideal I, denoted by $\mathcal{F}(I)$, is the extended complete fan obtained from the restricted Gröbner fan $\mathcal{F}^+(I)$ by adding a ray determined by the vector $(-1, \ldots, -1)$.

DEFINITION 3.11. For a homogeneous ideal I, the saturation of I, denoted by I^{sat} is the ideal $\{f \in S \mid \forall i, 1 \le i \le n, \exists n_i \in \mathbb{N} \text{ s.t. } x_i^{n_i} f \in I\}$. Using the ideal quotient, $I^{\text{sat}} = \bigcup_{k \ge 0}(I :$

$(x_0, \ldots, x_n)^k)$. I is said to be *m-saturated* if $(I)_j = (I^{\text{sat}})_j$ for $j \geq m$.

Facts on saturated ideals

(1) $\operatorname{Proj} S/I_1 = \operatorname{Proj} S/I_2 \Longleftrightarrow I_1^{\text{sat}} = I_2^{\text{sat}}$.
(2) $I_1^{\text{sat}} = I_2^{\text{sat}} \Longleftrightarrow \exists n \in \mathbb{N}$ s.t. $(I_1)_d = (I_2)_d \; \forall d \geq n$.
(3) If for any $J \supseteq I$, $J_d = I_d$ for all $d >> 0$, then $J = I \Longrightarrow I$ is saturated, that is, $I^{\text{sat}} = I$.
(4) Let X be the subscheme of $\mathbb{P}^n$ defined by I. Then I is saturated if I is the largest ideal of S which defines X.
(5) I is saturated if in the primary decomposition of I, the irrelevant ideal $(x_0, \ldots, x_n)$ does not occur as an associated prime.

Definition 3.12. Let $\mathcal{F}(I_1) := \{C_{11}, \ldots, C_{1k_1}\}$ and $\mathcal{F}(I_2) := \{C_{21}, \ldots, C_{2k_2}\}$ be two complete Gröbner fans. $\mathcal{F}(I_2)$ is said to coarser than $\mathcal{F}(I_1)$, denoted by $\mathcal{F}(I_1) \geq \mathcal{F}(I_2)$ if for all $i, 1 \leq i \leq k_1$ there is a $j, 1 \leq j \leq k_2$ such that $C_{1i} \subseteq C_{2j}$.

In the following, for more details on the stable Gröbner fan of an ideal, we refer to the paper D. Mall [4].

Notation 3.13. Let I be a homogeneous ideal of S.

(1) $I_{\geq d} := \bigoplus_{\geq d} I_d$
(2) ${}^rI := \bigoplus_{\geq m(I)+r} I_d$, where $m(I)$ is the minimum degree of polynomials in a minimal generating basis of I.

Remark 3.14. $\mathcal{F}(I) \geq \mathcal{F}({}^1I) \geq \mathcal{F}({}^2I) \geq \cdots \geq \mathcal{F}({}^rI) \geq \ldots$.

Theorem 3.15. *There exists a coarest fan in* $(\{\mathcal{F}(J) \mid J^{\text{sat}} = I^{\text{sat}}\}, \geq)$, *which is called the stable Gröbner fan of I and denoted by* $\mathcal{F}_{\text{stab}}(I)$.

Proof. Let I_1, I_2 be homogeneous ideals in the same saturation class. Then there exists an integer $d \in \mathbb{N}$ such that $(I_1)_{\geq d} = (I_2)_{\geq d}$. Then there are $r_1, r_2 \in \mathbb{N}$ such that ${}^{r_1}I_1 = {}^{r_2}I_2$. Since $\mathcal{F}(I)$ is a finite object and $\mathcal{F}({}^rI) \geq \mathcal{F}({}^{r+1}I)$, the following sequence $\mathcal{F}(I) \geq \mathcal{F}({}^1I) \geq \mathcal{F}({}^2I) \geq \cdots \geq \mathcal{F}({}^rI) \geq \ldots$ stops.

LEMMA 3.16. *Let* $Q(t) = \binom{t-\alpha_0+\ell}{\ell} + \cdots + \binom{t-\alpha_s+\ell-s}{\ell-s}$, *with* $0 \le s < \ell$ *and* $\alpha_0 \le \ldots, \alpha_s$, *the Hilbert polynomial of a saturated homogeneous ideal* I *of* S *and* $\prec$ *a term order. Then* $in_{\prec}(I)$ *is* α_s*-saturated.*

PROPOSITION 3.17. *Let* I *be a saturated homogeneous ideal of* $S = k[x_0, \ldots, x_n]$ *with the Hilbert polynomial* $Q(t)$ *given at Lemma 3.16. Then* $\mathcal{F}^{\alpha_s}(I) = \mathcal{F}_{stab}(I)$.

Proof. By Lemma 3.16, I and $in_{\prec}(I)$, $\prec \in \mathcal{TO}$, are α_s-saturated. Thus there are no $\prec_1, \prec_2 \in \mathcal{TO}$ and $m \in \mathbb{N}$, $m \ge \alpha_s$ such that $in_{\prec_1}(I_{\ge m}) = in_{\prec_2}(I_{\ge m})$. A jump in the sequence of fans $\{\mathcal{F}^r(I) \mid r \in \mathbb{N}\}$ comes from the collapse of Gröbner cones, hence it follows that $\mathcal{F}^{\alpha_s}(I) = \mathcal{F}_{stab}(I)$.

Let us denote $\mathcal{F}^d(I) := \mathcal{F}(I_{\ge d})$. Let $d_1 = m(I)$, and let d_2 be the smallest integer such that $\mathcal{F}(I) = \mathcal{F}^{d_1}(I) > \mathcal{F}^{d_2}(I)$, and d_3 be the smallest integer such that $\mathcal{F}^{d_2}(I) > \mathcal{F}^{d_3}(I), \ldots$.

Then we have a sequence of integers $(d_1, d_2, \ldots, d_s)$, which is called the *jump vector* of the homogeneous ideal I.

REMARK 3.18. Let I be a saturated ideal. The jump vector of I, $(d_1, d_2, \ldots, d_s)$ and the fan sequence of I, $S\mathcal{F}(I^{\text{sat}}) := (\mathcal{F}^{d_1}(I), \mathcal{F}^{d_2}(I), \ldots, \mathcal{F}^{d_s}(I))$ are invariants of the scheme Proj S/I. These invariants are dependent on the projective linear change of coordinates.

Question. Are there any relations between known quantities coming from an ideal I, for instance the regularity of I and the above invariants; the jump vector and the fan sequence of the saturation of I. What can we say about the jump vector and the fan sequences of I under generic projective linear coordinate changes.

4. Toric Ideals

Let us fix a subset $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of $\mathbb{Z}^d$. Similarly for the polynomial ring, we identify each vector $\mathbf{a}_i$ with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $k[\mathbf{t}^{\pm}] = k[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$ under the monoid isomorphism $\mathbf{log} : \mathbb{Z}^d \longrightarrow k[\mathbf{t}^{\pm}]$ defined by $\mathbf{log}(\mathbf{a}) = \mathbf{t}^{\mathbf{a}}$. By considering the set $\mathcal{A}$ as a $d \times n$ matrix, we may define a semigroup homomorphism $\pi_{\mathcal{A}} : \mathbb{N}^n \longrightarrow \mathbb{Z}^d$, $\pi_{\mathcal{A}}(\mathbf{u}) = \pi_{\mathcal{A}}(u_1, \ldots, u_n) = u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n = \mathcal{A}\mathbf{u}$. The map $\pi_{\mathcal{A}}$ lifts to a homomorphism of monoid algebra via

$$\tilde{\pi}_{\mathcal{A}} : k[\mathbf{x}] \longrightarrow k[\mathbf{t}^{\pm}],\ x_j \longmapsto \mathbf{t}^{\mathbf{a}_j}.$$

DEFINITION 4.1. The *toric ideal* of $\mathcal{A}$, denoted by $\mathcal{I}_{\mathcal{A}}$, is the kernel of the above homomorphism $\tilde{\pi}_{\mathcal{A}}$.

LEMMA 4.2. *(B. Sturmfels [7])* $\mathcal{I}_{\mathcal{A}} = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u} - \mathbf{v} \in ker(\pi_{\mathcal{A}}) \rangle$.

By Lemma 4.2, we note that a toric ideal is a binomial ideal, an ideal generated by polynomials with at most two terms of the same degree, thus a homogeneous ideal.

For each vector $\mathbf{u} \in \mathbb{N}^n$, let $\mathbf{u}^+$ and $\mathbf{u}^-$ be the unique vectors in $\mathbb{N}^n$ of disjoint supports such that $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$.

COROLLARY 4.3. $\mathcal{I}_{\mathcal{A}} = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \mid \mathbf{u} \in ker(\pi_{\mathcal{A}}) \rangle$.

The *initial complex* $\triangle_{\prec}(I)$ of an ideal I of $S = k[x_1, \ldots, x_n]$ with respect to a term order $\prec$ is the simplicial complex on the set $\{1, \ldots, n\}$ as follows: A subset $F \subseteq \{1, \ldots, n\}$ is a face of $\triangle_{\prec}(I)$ if there is no polynomial $f \in I$ whose the initial monomial $in_{\prec}(f)$ has the support F.

We now identify the set of vectors $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ with the index set $\{1, \ldots, n\}$. A *subdivision* of $\mathcal{A}$, denoted by $\triangle_{\mathcal{A}}$, is a collection of subsets σ of $\{1, \ldots, n\}$ such that the convex cones $\mathcal{C}(\sigma) = \mathcal{C}(\{\mathbf{u}_{i_j} | i_j \in \sigma\})$ form a polyhedral fan with support $\mathcal{C}(\mathcal{A})$. A subdivision $\triangle_{\mathcal{A}}$ of $\mathcal{A}$ is called a *triangulation* if for each cell σ, the convex cone $\mathcal{C}(\sigma)$ is simplicial.

For a generic vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$, define a triangulation $\triangle_{\mathbf{u}}$ of $\mathcal{A}$ as follows: A subset $\{i_1, \ldots, i_k\}$ is a face of $\triangle_{\mathbf{u}}$ if there exists a vector $\mathbf{c} = (c_1, \ldots, c_d) \in \mathbb{R}^d$ such that

$$\mathbf{a}_j \cdot \mathbf{c} = u_j \quad \text{if} \quad j \in \{i_1, \ldots, i_k\} \quad \text{and}$$

$$\mathbf{a}_j \cdot \mathbf{c} < u_j \quad \text{if} \quad j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}.$$

A triangulation $\triangle_{\mathcal{A}}$ of $\mathcal{A}$ is called *regular* if $\triangle_{\mathcal{A}} = \triangle_{\mathbf{u}}$ for some $\mathbf{u} \in \mathbb{R}^n$.

THEOREM 4.4. *(B. Sturmfels [7]) If a term order $\prec$ is represented by an array of vectors $\mathcal{U} = (\mathbf{u}_i, \ldots, \mathbf{u}_s)$ with $\mathbf{u}_1 = \mathbf{u} \in \mathbb{R}^n$, then $\triangle_{\prec}(I_{\mathcal{A}}) = \triangle_{\mathbf{u}}$, that is, the regular triangulations of $\mathcal{A}$ are the initial complexes of the toric ideal $I_{\mathcal{A}}$.*

A triangulation $\triangle_{\mathcal{A}}$ of $\mathcal{A}$ is called *unimodular* if $Vol(\sigma) = 1$ for every maximal simplex σ in $\triangle_{\mathcal{A}}$.

PROPOSITION 4.5. *(B. Sturmfels [7]) The initial ideal $in_{\prec}(I_{\mathcal{A}})$ is square-free if and only if the corresponding regular triangulation $\triangle_{\prec}$ of $\mathcal{A}$ is unimodular.*

The matrix $\mathcal{A}$ is called *unimodular* if all triangulations $\triangle_{\mathcal{A}}$ of $\mathcal{A}$ are unimodular.

COROLLARY 4.6. *A matrix is unimodular if and only if all initial ideals of the toric ideal $I_{\mathcal{A}}$ are square-free.*

LEMMA 4.7. *For an ideal I, $I^{\text{sat}} \subseteq \sqrt{I}$.*

Proof. Let $f \in I^{\text{sat}}$, $f \notin I$. Then for each i, $0 \leq i \leq n$, there exists $n_i > 0$ such that $x_i^{n_i} f \in I$. Let $N = \max_i \{n_i\}$. Let $f = \sum_{i=1}^{M} c_i \mathbf{x}^{\mathbf{a}_i}$. Then

$$\begin{aligned} & f^{NM+1} \\ =& f^{NM} f \\ =& \Big(\sum_{i=1}^{M} c_i \mathbf{x}^{\mathbf{a}_i}\Big)^{MN} f \\ =& \sum_{k_1+\cdots+k_M=MN} \frac{(MN)!}{k_1! \ldots k_M!} (c_1 \mathbf{x}^{\mathbf{a}_1})_1^k \ldots (c_M \mathbf{x}^{\mathbf{a}_M})^{k_M} f. \end{aligned}$$

There exists j, $1 \leq j \leq M$, such that $k_j \geq N$. (Otherwise $k_1 + \cdots + k_M < MN$.) Then $(c_j \mathbf{x}^{\mathbf{a}_j})^{k_j} f \in I$. This is true for each set $\{k_1, \ldots, k_M\}$ satisfying the summation, $k_1 + \cdots + k_M = MN$. Hence $f^{MN+1} \in I$, so $f \in \sqrt{I}$.

We define a *unimodular toric ideal* to be a toric ideal $I_{\mathcal{A}}$ with a unimodular matrix $\mathcal{A}$. The following statement was just briefly mentioned at [4] without details.

PROPOSITION 4.8. *For unimodular toric ideals $I_{\mathcal{A}}$, $\mathcal{F}_{\mathrm{stab}}(I_{\mathcal{A}}) = \mathcal{F}(I_{\mathcal{A}})$, that is, the stable Gröbner fan of $I_{\mathcal{A}}$ is identical to the ordinary Gröbner fan of $I_{\mathcal{A}}$. Especially there is no jump for unimodular toric ideals $I_{\mathcal{A}}$.*

Proof. Let $I_{\mathcal{A}}$ be a unimodular toric ideal. Then by Corollary 4.6, $in_{\prec}(I_{\mathcal{A}})$ is square-free, that is, $\sqrt{in_{\prec}(I_{\mathcal{A}})} = in_{\prec}(I_{\mathcal{A}})$. From Lemma 4.7,

$$in_{\prec}(I_{\mathcal{A}})^{\mathrm{sat}} \subseteq \sqrt{in_{\prec}(I_{\mathcal{A}})} = in_{\prec}(I_{\mathcal{A}}).$$

Thus we conclude that initial ideals of unimodular toric ideals are saturated. Then from the argument in the proof of Theorem 3.15, our conclusion follows.

REMARK 4.9. B. Huber and R. Thomas [3] gives an algorithm for the computation of Gröbner Fans of Toric Ideals.

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