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COPRIMELY PACKED RINGS (II)

YONG HWAN CHO

Dept. of Mathematics Education,
Chonbuk National University, Chonju, Chonbuk 561-756, Korea.

Abstract In this paper we show that in case of Noetherian ring $R(1)$ if R is coprimely packed then $R((X))$ is coprimely packed and (2) if $Max(R)$ is coprimely packed then so is $MaxR((X))$.

1. Introduction

Let R be a commutative ring with identity and I an ideal of R. Then I is said to be *coprimely packed by prime ideals* if whenever I is coprime to each element of a family of prime ideals of R, I is not contained in the union of prime ideals in the family. If every ideal of R is coprimely packed, then we say R is coprimely packed. $Max(R)$ will denote the set of maximal ideals of R. For any maximal ideal M of R,$\Omega(M)$ will denote the set $Max(R) - \{M\}$. We say that $Max(R)$ is coprimely packed if each maximal ideal of R is not contained in the union of the other maximal ideals. Let R be the polynomial ring over R and S be the set of all polynomial $f = c_0 + c_1x + \cdots + c_nx^n \in R[X]$ such that $(c_0, c_1, \cdots, c_n) = R$. Let $S^{-1}R[X] = R(X)$. Similarly,let S be the set of all $f \in R[[X]]$ such that $A_f = R$ where A_f is the ideal of R generated by the coefficients of f. Then S is a multiplicative system in $R[[X]]$.Denote the quotient ring $S^{-1}R[[X]]$ by $R((X))$. V.Erdoğdu proved that $Max(R)$ is coprimely packed if only if $MaxR[[X]]$ is coprimely packed([3]) and that if R is coprimely packed then so is $R(X)$ ([2]). In this paper we prove that in case of Noetherian ring R, if $Max(R)$ is coprimely packed then

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$MaxR((X))$ is coprimely packed and if R is coprimely packed then $R((X))$ is coprimely packed.

2. Coprimely packedness

PROPOSITION 2.1. *Let J be an ideal of R contained in the Jacobson radical of R. Then R is coprimely packed if and only if the same is so of R/J.*

Proof. Suppose that R/J is coprimely packed. Let I be any ideal of R and X any non- empty subset of $SpecR$ such that $I+P=R$, for all $P \in X$. Then clearly neither I nor any P in X is contained in the Jacobson radical of R. Since R/J is coprimely packed, and $\bar{I}+\bar{P}=R/J$ in R/J (where bar denote the image of ideals of R in R/J),we have $\bar{I} \not\subseteq \cup_{P \in X} \bar{P}$ in R/J. But then it follows that $I \not\subseteq \cup_{P \in X} P$ in R. Therefore R is coprimely packed. Conversely,suppose that R is coprimely packed. Let I/J be any ideal of R/J and Y a non-empty subset of $Spec(R/J)$ such that $I/J+P/J=R/J$ for all $P/J \in Y$ Since $J \subset I, P$, $I+P=R$ for all $P/J \in Y$. If $I/J \subseteq \cup_{P/J \in Y} P/J$ then $I \subseteq \cup_{P/J \in Y} P$. But this is impossible because R is coprimely packed. Therefore $I/J \not\subseteq \cup_{P/J \in Y} P/J$.Thus R/J is coprimely packed.

PROPOSITION 2.2. *([3]) Suppose that every prime ideal of R is coprimely packed by the set of maximal ideals of R. Then R is coprimely packed.*

THEOREM 2.3. *([3]) Let R be a ring.Then $Max(R)$ is coprimely packed if and only if $MaxR[[X]]$ is coprimely packed.*

If I is an ideal of R then $I[[X]]=\{f=\sum_{i=0}^{\infty} a_i X^i | a_i \in I$ for all $i\}$ Then $IR[[X]]$ is an ideal of $R[[X]]$. In general, $IR[[X]] \neq I[[X]]$.

PROPOSITION 2.4. *([1]) Let R be a Noetherian ring and I an ideal of R. Then $I[[X]]=IR[[X]]$*

PROPOSITION 2.5. *Let R be a Noetherian ring.Then for all ideal I of R, $I[[X]]R((X))=IR((X))$*

Proof. Let f be any power series in $I[[X]]R((X))$. Then $f= \sum_{i=1}^{n} f_i \frac{g_i}{h_i}$ where $f_i \in I[[X]], g_i \in R[[X]]$ and $h_i \in S$, multiplicative system of $R[[X]]$. Put $k_i = f_i g_i$. Then $k_i \in I[[X]]$. By Proposition 2.4, $k_i = \sum_{j=1}^{t} a_{ij} s_{ij}$ where $a_{ij} \in I, s_{ij} \in R[[X]]$. Therefore, $f = \sum_{i=1}^{n} \frac{k_i}{h_i} = \sum_{i=1}^{n} (\sum_{j=1}^{t} a_{ij} \frac{s_{ij}}{h_i})$ Thus $I[[X]]R((X)) \subseteq IR((X))$. The converse is trivial.

PROPOSITION 2.6. *Let S be the set of all $f \in R[[X]]$ such that $A_f = R$. Then*

(1) *S is a multiplicative system in $R[[X]]$ and $S = R[[X]] - \cup_{\alpha \in I} M_\alpha[[X]]$ where $\{M_\alpha\}_{\alpha \in I}$ is the set of maximal ideals of R.*

(2) *If $\{M_\alpha\}_{\alpha \in I}$ is the maximal ideals of R then $\{M_\alpha[[X]]R((X))\}_{\alpha \in I}$ is the set of maximal ideals of $R((X))$. Specially, if R is a Noetherian ring then $\{M_\alpha R((X))\}_{\alpha \in I}$ is the maximal ideals of $R((X))$.*

Proof. (1) Let $f \in R[[X]]$. Then $A_f = R$ if and only if $A_f \nsubseteq M_\alpha$ for each $\alpha \in I$. Futhermore,Since $M_\alpha[[X]]$ is prime, S is a multiplicative system.

(2)Let P be a prime ideal of $R[[X]]$ such that $P \subseteq \cup_{\alpha \in I} M_\alpha[[X]]$. and let $A = \{c_i \in R | c_i$ is a coefficient of $f, f \in P\}$. Then A is an ideal of R. Since $1 \notin A$ A is a proper ideal. Thus $A \subseteq M_\alpha$ for some $\alpha \in I$. For any $f \in P$, all coefficients of f are in A and are contained in M_α. Therefore $P \subseteq M_\alpha[[X]]$. By Proposition 4.8 of [4], $\{M_\alpha[[X]]R((X))\}_{\alpha \in I}$ is the set of maximal ideals of $R((X))$. Last part of (2) comes from Proposition 2.5.

PROPOSITION 2.7. *([6]) Let R be a Noetherian ring with an identity and S the set of all $f \in R[[X]]$ such that $A_f = R$. Then*

(1) *S is a multiplicative system in $R[[X]]$ consisting entirely of regular elements in $R[[X]]$.*

(2) *If Q is a P-primary ideal of R then $QR((X))$ is a $PR((X))$-primary ideal of $R((X))$ and $QR((X)) \cap R = Q$.*

THEOREM 2.8. *Let R be a Noetherian ring with unity. Then*

(1) *If $Max(R)$ is coprimely packed then $MaxR((X))$ is coprimely packed.*

(2) *If $MaxR[[X]]$ is coprimely packed then $MaxR((X))$ is coprimely packed.*

Proof. (1) Suppose that $MaxR((X))$ is not coprimely packed and let M be a maximal ideal of R such that $M' \subseteq \cup_{N' \in \Omega(M')} N'$. Note that $M' = MR((X))$ is a maximal ideal of $R((X))$ and any maximal ideal N' of $R((X))$ is of the form $NR((X))$ for some maximal ideal N of R (Proposition 2.6). Therefore, $MR((X)) \subseteq \cup_{N \in \Omega(M)} NR((X))$. So $MR((X)) \cap R \subseteq \{\cup_{N \in \Omega(M)} NR((X))\} \cap R = \cup_{N \in \Omega(M)} \{NR((X)) \cap R\}$. By Proposition 2.7, $M \subseteq \cup_{N \in \Omega(M)} N$ and $Max(R)$ is not coprimely packed.

(2) By (1) and Theorem 2.3, it is clear.

THEOREM 2.9. *Let R be a ring. Then If $Max(R)$ is coprimely packed then $R(X)$ is coprimely packed.*

Proof. By Theorem 14.1 ([5]) we can easily prove this theorem.

THEOREM 2.10. *Let R be a Noetherian ring with unity.If R is coprimely packed then $R((X))$ is coprimely packed.*

Proof. To prove that $R((X))$ is coprimely packed it is enough to show that every prime ideal of $R((X))$ is coprimely packed by the set of maximal ideals of $R((X))$. (Proposition 2.2) Let $\mathcal{P}$ be any prime ideal of $R((X))$ and Γ be any non-empty subset of $MaxR((X))$ such that $\mathcal{P} + \mathcal{M} = R((X))$ for all $\mathcal{M} \in \Gamma$. We know that $\mathcal{P} = PS^{-1}R[[X]]$ for some prime ideal P of $R[[X]]$ such that $P \cap S = \emptyset$ and for each $\mathcal{M} \in \Gamma$ $\mathcal{M} = M[[X]]S^{-1}R[[X]]$ such that $M[[X]] \cap S = \emptyset$ where M is a maximal ideal of R. Thus we have $PS^{-1}R[[X]] + MS^{-1}R[[X]] = (P + M[[X]])S^{-1}R[[X]] = R((X))$ (Proposition 2.5),for all $M \in \Gamma_*$,where $\Gamma_* = \{M | MR((X)) \in \Gamma\} \subseteq Max(R)$. Hence $(P + M[[X]]) \cap S \neq \emptyset$. Let A be the ideal of R consisting of the coefficients of the power series in P. If $A \subseteq M$ for some $M \in \Gamma_*$, then $P \subseteq M[[X]]$ and so $M[[X]]S^{-1}R[[X]] = MR((X)) = R((X))$, a contradiction. Thus $A \not\subseteq M$ for all $M \in \Gamma_*$. Since R is coprimely packed, $A \not\subseteq \cup_{M \in \Gamma_*} M$. Let $c \in A$ with $c \notin M$ for all $M \in \Gamma_*$ .Let f be a power series in P having c as some coefficient. Since $f \notin \cup_{M \in \Gamma_*} M[[X]]$, we see that $\mathcal{P} \not\subseteq \cup_{\mathcal{M} \in \Gamma} \mathcal{M}$, and so $R((X))$ is coprimely packed.

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