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# MODULES OVER ASSOCIATED GRADED RINGS

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## 1. Introduction

Put for simplicity $R = \mathbb{Z}\ [C(p)]$, $R_1 = \mathbb{Z}$, $R_2 = \mathbb{Z}[\theta]$, where $\theta$ is a primitive $p$th root of 1, $p_1 = p$, $p_2 = 1 - \theta$, $\bar{R} = \mathbb{Z}/p\ \mathbb{Z}$. Both $R_1, R_2$ are dedekind domains and project onto $\bar{R}$; let $v_1, v_2$ denote the canonical homomorphism of $R_1, R_2$ respectively, onto $\bar{R}$, so $\mathrm{Ker} v_i = P_i = R_i p_i$ for all $i = 1, 2$. Then $R$ is (isomorphic to) the pullback

$$\{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\} \tag{1}$$

(denoted it by $(R_1 \overset{v_1}{\longrightarrow} \bar{R} \overset{v_2}{\longleftarrow} R_2))$ via $g \to (1, \theta)$ where $g$ is a fixed generator of $C(p)$ (see [7]). Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \oplus P_2$, $R/P \cong R_1/P_1 \cong R_2/P_2 \cong \bar{R}$, and $P_1 P_2 = P_2 P_1 = 0$. Furthermore, for $i \neq j$, the sequence $0 \to P_i \to R \to R_j \to 0$ is an ex act sequence of $R$-modules (see [6]).

Let $R$ be the pullback ring as described in (1). Assume furthermore that $G(R)$ is the associated graded ring formed with respect to $P$. Our aim here is to establish a bijection between isomorphism classes of indecomposable finitely generated $R$-modules, for the pullback ring $R$, and isomorphism classes of indecomposable modules over the associated graded ring $G(R)$ of $R$. Also we give a description of some indecomposable pure-injective modules over $G(R)$.

An $R$-module $S$ is called separated if there exists an $R_i$-module $S_i$, $i = 1, 2$, such that $S$ is a submodule of $S_1 \oplus S_2$ ( made into an $R$-module by $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$). Alternatively, $S$ is separated if it is a pullback, in fact, $S = (S/P_2 S \to S/PS \leftarrow$

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$S/P_1S$) [6, lemma 3.3]. Also $S$ is separated if and only if $P_1S \cap P_2S = 0$ [6, lemma 2.9].

Let $R$ be any ring, and let $I$ be an $R$-ideal with $R \neq I$ and $M$ an $R$-module. The ideals $I^n$ (where we set $I^0 = R$) form a descending chain $I^0 \supset I \supset I^2 \supset \ldots$ of ideals in $R$, and the modules $I^nM$ form a descending chain of submodules of $M$. Define the **associated graded ring** $G(R)$ to be the additive group $\bigoplus_{n\geq 0} I^i/I^{i+1}$ made into a ring by defining multiplication

$$(r_i + I^{i+1})(s_j + I^{j+1}) = r_is_j + I^{i+j+1}, \quad \text{where} \quad r_i \in I^i, s_j \in I^j.$$

Similarly, the **associated graded module** is $\bigoplus_{n\geq 0} I^iM/I^{i+1}M$, made into a module over $G(R)$ by means of scalar multiplication

$$(r_i+I^{i+1})(x_j+I^{j+1}M) = r_ix_j+I^{i+j+1}M \text{ where } r_i \in I^i, x_j \in I^jM.$$

Of course, there is a functor $G$ sending the $R$-module $M$ to the $G(R)$-module $G(M)$. For if $f : M \to N$ is a morphism, then for each $i$, $f$ induces canonical mappings $f_i : I^iM/I^{i+1}M \to I^iN/I^{i+1}N$, given by $f_i(x+I^{i+1}M) = f(x)+I^{i+1}N$ for $x \in I^iM$. Putting $G(f) = \bigoplus_{i\geq 0} f_i$ defines a morphism of $G(R)$-modules.

A module $I$ is pure-injective if and only if any system of equations in $I$ which is finitely solvable in $I$, has a global solution in $I$ [9, theorem 2.8], in fact, an $R$-module $I$ is called pure-injective if it satisfies the equivalent conditions of the [5, theorem 7.1].

THEOREM 1.1. *Let $R$ be the pullback ring as described in (1). Then the list of indecomposable pure-injective separated $R$-modules are:*

*(1) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2))$ where $E(R_i/P_i)$ is the injective hull of $R_i/P_i$ for $i = 1, 2$;*

*(2) $S = (Q(R_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow Q(R_2))$ where $Q(R_i)$ is the field of fractions of $R_i$ for $i = 1, 2$;*

*and, for all positive integers $n$, $m$,*

*(3) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$;*

*(4) $S = (\hat{R}_1 \to \bar{R} \leftarrow R_2/P_2^m)$;*

*(5) $S = (R_1/P_1^n \to \bar{R} \leftarrow \hat{R}_2)$;*

*(6) $S = (\hat{R}_1 \to \bar{R} \leftarrow \hat{R}_2)$*

*(where, $\hat{R}_i$ is the completion of $R_i$ at $P_i$ for $i = 1, 2$)*

*Proof.* See [10, lemma 4, 5, 6].

## 2. pure-injective modules over $G(R)$

PROPOSITION 2.1. *Let $R$ be a complete local ring with maximal ideal $P$. Suppose that $M$ is an $R$-module. Assume furthermore that $G(R)$ is the associated graded ring formed with respect to $P$. Then if $M$ is a pure-injective $R$-module, then $G(M)$ is pure-injective as a $G(R)$-module.*

*Proof.* If $M = PM$ then $G(M) = 0$, so it is pure-injective since every finite module is pure-injective. Supppse that $M \neq PM$. First, we prove that if $I$ is an arbitrary index set then $G(M^{[I]})$ is canonically isomorphic to a submodule $B$ of $G(M^I)$ and $G(M^I/M^{[I]}) \cong G(M^I)/B$. In order to prove this statement, it is enough to show that for each $n$,

$$G(M^I/M^{[I]})_n \cong (P^nM^I/P^{n+1}M^I)/((P^{n+1}M^I + P^nM^{[I]})/P^{n+1}M^I).$$

We have $P^n(M^I/M^{[I]}) = (P^nM^I + M^{[I]})/M^{[I]}$. Therefore

$$G(M^I/M^{[I]})_n = P^n(M^I/M^{[I]})/P^{n+1}(M^I/M^{[I]})$$

is canonically isomorphic to

$$(P^nM^I/P^{n+1}M^I)/(P^nM^I \cap (P^{n+1}M^I + M^{[I]})/P^{n+1}M^I).$$

Thus $G(M^I/M^{[I]})_n \cong (P^nM^I/P^{n+1}M^I)/((P^{n+1}M^I + (P^nM^I \cap M^{[I]}))/P^{n+1}M^I)$. Since for each $n$, $P^nM^I \cap M^{[I]} = P^nM^{[I]}$, we obtain $G(M^I/M^{[I]})_n$ is isomorphic to $(P^nM^I/P^{n+1}\ M^I)/((P^{n+1}\ M^I + P^n\ M^{[I]})/P^{n+1}\ M^I)$. Moreover we have $(P^{n+1}\ M^I + P^n\ M^{[I]})/P^{n+1}\ M^I \cong P^n\ M^{[I]}/(P^{n+1}\ M^I \cap P^n\ M^{[I]} = G(M^{[I]})_n$. Since $P^{n+1}\ M^I \cap P^n\ M^{[I]} = P^{n+1}M^{[I]}$, as required.

Next, suppose that $M$ is a pure-injective $R$-module. Since $M$ is pure-injective, by [5, theorem 7.1 (vii)], the diagonal embedding $\triangle : M \to M^I/M^{[I]}$ is split: that is, there is a mapping $h$ :

$M^I/M^{[I]} \to M$ such that $h\triangle = 1_M$. Since $G$ is a functor and by above statement, there are the mappings

$$G(\triangle) : G(M) \to G(M^I/M^{[I]}) = G(M^I)/G(M^{[I]})$$
$$= G(M)^I/G(M)^{[I]}$$

and $G(h) : G(M)^I/G(M)^{[I]} \to M$ such that $G(h)G(\triangle) = 1_{G(\triangle)}$, as required by [5, theorem 7.1 (vii)].

LEMMA 2.2. *Let $R$ be a complete local ring with maximal ideal $P$. Suppose that $G(R)$ is the associated graded ring formed with respect to $P$. Assume furthermore that the $R$-module $M$ is a Hausdorff space for its $P$-topology. If $M$ is a non-zero indecomposable $R$-module with $dim_{R/P}(M/PM) < \infty$, $M \neq PM$ then $G(M)$ is non-zero and indecomposable as a $G(R)$-module. Conversely, if $G(M)$ is indecomposable with $G(M) \neq 0$, then $M$ is indecomposable.*

*Proof.* By assumption and [13, ch VIII theorem 7, lemma 2], $M$ is finitely generated over $R$. Let $M$ be an indecomposable $R$-module, and let $G(M) = N_1 \oplus N_2$. Then by [1, theorem 5.1], $M = M_1 \oplus M_2$ for some $R$-modules $M_1$ and $M_2$ with $N_1 = G(M_1)$ and $N_2 = G(M_2)$. Thus either $M_1 = 0$ or $M_2 = 0$. It follows that either $N_1 = 0$ or $N_2 = 0$. Similarly, if $0 \neq G(M)$ is indecomposable, then $M$ is indecomposable since $M$ is a Hausdorff space.

Let $R$ be the pullback ring as described in (1). By [10, lemma 2], If $\hat{R}_i$ are the completion of $R_i$ at $P_i$, $i = 1, 2$, and $S$ is a pure-injective separated $R$-module then $S$ is a module over the pullback ring

$$\hat{R} = (\hat{R}_1 \to \bar{R} \leftarrow \hat{R}_2). \tag{2}$$

Therefore we can now assume that $R$ is the pullback of two complete local rings $R_1, R_2$ with maximal ideals $P_1, P_2$ respectively and $R/P \cong R_1/P_1 \cong R_2/P_2 \cong \bar{R}$ a field where $P = P_1 \oplus P_2$. Assume furthermore that $G(R)$ is the associated graded ring formed with respect to $P$. Then by [1, proposition 5.2], $G(R) = (G(R_1) \to \bar{R} \leftarrow G(R_2))$, the jacobson radical $J(G(R))$ is equal to $G(P)$, and $G(R)/G(P) \cong \bar{R}$.

PROPOSITION 2.3. *Let $R$ be the pullback ring as described in (2). If $S$ is a separated $R$-module with $S \neq PS$, then $G(S)$ is separated as an $G(R)$-module.*

*Proof.* Let $S = (S/P_2S = S_1 \to \bar{S} = S/PS \leftarrow S_2 = S/P_1S)$ be a separated $R$-module. We show that $G(S) = (G(S_1) \to \bar{S} \leftarrow G(S_2))$ is a separated $G(R)$-module. To see this, it is enough to show that $G(S/P_iS) = G(S)/G(P_iS) = G(S)/G(P_i)G(S)$ since $G$ is a functor (note that $P\bar{S} = P_i\bar{S} = 0$, $G(\bar{S}) = \bar{S}$). By definition of the functor $G$ we see that $G(P_iS) = G(P_i)G(S)$. Thus it suffices to prove that for each $n$,

$$L = G(S/P_1S)_n \cong (P^nS/P^{n+1}S)/(P^nP_1S/P^{n+1}P_1S)$$

We have $P^n(S/P_1S) = (P^nS + P_1S)/P_1S$. Therefore $L$ is canonically isomorphic to

$$(P^nS/P^{n+1}S)/(P^nS \cap (P^{n+1}S + P_1S)/P^{n+1}S).$$

Thus $L \cong (P^nS/P^{n+1}S)/(P^{n+1}S + (P^nS \cap P_1S))/P^{n+1}S$. Since $P^nS \cap P_1S = P^nP_1S$, $L \cong (P^nS/P^{n+1}S)/(P^{n+1}S + P^nP_1S)/P^{n+1}S$. So

$$L \cong (P^nS/P^{n+1}S)/P^nP_1S/(P^{n+1}S \cap P^nP_1S) \cong G(S)_n/G(P_1S)_n$$

since $P^{n+1}S \cap P^nP_1S = P^{n+1}P_1S$. Similarly, $G(S/P_2S) = G(S)/G(P_2S)$, as required.

THEOREM 2.4. *Let $R$ be the pullback ring as described in (2).*

*(i) Let $S$ be one of the separated modules as described in 1) - 2) of the theorem 1.1. Then $G(S) = 0$ which is pure-injective as an $G(R)$-module.*

*(ii) Let $S$ be one of the separeted $R$-modules as described in 3) - 6) of the thorem 1.1. Then $G(S)$ is an indecomposable pure-injective and separated as an $G(R)$-module.*

*Proof.* (i) This follows from 2.1.

(ii) Let $S$ be one of the separated $R$-modules as described in 3) - 6) of the theorem 1.1. Then by 2.2, 2.1, and 2.3, $G(S)$ is indecomposable pure-injective and separated as an $G(R)$-module.

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