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CERTAIN METHODS FOR PROVING TWO RESULTS CONTIGUOUS TO KUMMER'S SECOND THEOREMS

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Abstract The object of this note is to provide various methods for proving two results contiguous to Kummer's second theorem by using the classical Gauss's summation theorem and contiguous relations.

1. Introduction and Preliminaries

The generalized hypergeometric function with p-numerator and

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q-denominator parameters is defined by

(1.1)
$$
{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};z\right] = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n z^n}{(\beta_1)_n\ldots(\beta_q)_n n!}
$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined by

$$
(\alpha)_0 = 1 \quad \text{and} \quad (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \quad (n = 1, 2, \cdots)
$$

for any complex number α.

From the theory of differential equations, Professor Kummer [3] established the following very interesting and useful results, which are in the literature, known as the Kummer's first and second theorems:

$$
e^{-x}\times {}_1F_1(\alpha;\rho;x) = {}_1F_1(\rho-\alpha;\rho;-x) \tag{1.2}
$$

and

$$
e^{-x}\times {}_1F_1(\alpha;2\alpha;2x) = {}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{4}). \tag{1.3}
$$

Later on, Professor Bailey [1] established the result (1.3) in the form

$$
e^{-x/2}\times {}_1F_1(\alpha;2\alpha;x) = {}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{16}) \tag{1.4}
$$

by making use of Gauss's second summation theorem [2]:

$$
{}_2F_1\left[\begin{matrix}a,b;\\ \frac{1}{2}(a+b+1);\end{matrix}\frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}. \tag{1.5}
$$

Very recently Rathie and Choi [9] derived the result (1.3) in the form

$$e^x \times {}_1F_1(-;\alpha+\frac{1}{2};\frac{x^2}{4}) = {}_1F_1(\alpha;2\alpha;2x) \tag{1.6}$$

by using classical Gauss's summation theorem [2]:

$$ {}_2F_1\left[\begin{matrix} a,b\,; \\ c\,; \end{matrix} 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.7}$$

provided $\mathrm{Re}(c-a-b)>0$.

In 1995, Rathie and Nagar [11] established the following two results contiguous to (1.4):

$$\begin{aligned} &e^{-x/2} \times {}_1F_1(\alpha;2\alpha+1;x) \\ &= {}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{16}) - \frac{x}{2(2\alpha+1)}\,{}_0F_1(-;\alpha+\frac{3}{2};\frac{x^2}{16}) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} &e^{-x/2} \times {}_1F_1(\alpha;2\alpha-1;x) \\ &= {}_0F_1(-;\alpha-\frac{1}{2};\frac{x^2}{16}) + \frac{x}{2(2\alpha-1)}\,{}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{16}) \end{aligned} \tag{1.9}$$

by employing two summation formulas [4] contiguous to Gauss's second summation theorem:

(1.10)

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} a,b \\ \frac{1}{2}(a+b+2) \end{matrix};\frac{1}{2}\right] \\ &= \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)}{a-b}\left\{\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})} - \frac{1}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})}\right\} \end{aligned}$$

and

$$
(1.11)\qquad {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b) \end{matrix}; \frac{1}{2}\right] = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b)\Big[\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)} + \frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}\Big].
$$

On the other hand, from the theory of differential equations, Professor Preece [5] established the following interesting and useful identity involving square of a generalized hypergeometric series:

$$
(1.12)\qquad \{{}_1F_1(\alpha; 2\alpha; x)\}^2 = e^x \times {}_1F_2(\alpha; \alpha+\frac{1}{2}, 2\alpha; \frac{x^2}{4}).
$$

Recently, Rathie [7] has given a very short proof of (1.12) and obtained two results contiguous to (1.12).

Later on, Rathie and Choi [8] derived the following two results closely related to (1.12) by the same short method developed by Rathie [7]:

$$
(1.13)\qquad \begin{aligned} \{{}_1F_1(\alpha; 2\alpha+1; x)\}^2 = e^x\Big\{ & {}_1F_2(\alpha; \alpha+\frac{1}{2}, 2\alpha; \frac{x^2}{4}) \\ & - \frac{x}{2\alpha+1}\, {}_1F_2(\alpha+1; \alpha+\frac{3}{2}, 2\alpha+1; \frac{x^2}{4}) \\ & + \frac{x^2}{4(2\alpha+1)^2}\, {}_1F_2(\alpha+1; \alpha+\frac{3}{2}, 2\alpha+2; \frac{x^2}{4})\Big\} \end{aligned}
$$

and

$$
(1.14)\qquad \begin{aligned} \{{}_1F_1(\alpha; 2\alpha-1; x)\}^2 = e^x\Big\{ & {}_1F_2(\alpha-1; \alpha-\frac{1}{2}, 2\alpha-2; \frac{x^2}{4}) \\ & + \frac{x}{2\alpha-1}\, {}_1F_2(\alpha; \alpha+\frac{1}{2}, 2\alpha-1; \frac{x^2}{4}) \\ & + \frac{x^2}{4(2\alpha-1)^2}\, {}_1F_2(\alpha; \alpha+\frac{1}{2}, 2\alpha; \frac{x^2}{4})\Big\}. \end{aligned}
$$

The following series identity [6, p. 56], and a result for the Pochhammer symbol [6, p. 59], and the well known Bailey's formula [2, p. 245] will also be required in our present investigations:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \tag{1.15}$$

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k} \quad (0 \le k \le n), \tag{1.16}$$

and

$$\begin{aligned} &{}_0F_1(-;\rho;x) \times {}_0F_1(-;\sigma;x) \\ &= {}_2F_3\left[\begin{matrix}\frac{1}{2}(\rho+\sigma), \frac{1}{2}(\rho+\sigma-1)\,; \\ \rho, \sigma, \rho+\sigma-1\,;\end{matrix}\; 4x\right], \end{aligned} \tag{1.17}$$

which, for $\rho = \sigma$, reduces to

$$\{{}_0F_1(-;\rho;x)\}^2 = {}_1F_2(\rho - \frac{1}{2}; \rho, 2\rho-1; 4x). \tag{1.18}$$

The aim of this note is to prove the results (1.8) and (1.9) by using (i) classical Gauss's summation theorem (1.6), (ii) contiguous relations, and (iii) results (1.13) and (1.14) contiguous to Preece's identity (1.12).

2. Proofs of (1.8) and (1.9)

First Method : Use of Gauss's theorem (1.7).

In order to prove the result (1.8), it is easy to see that the result (1.8) can also be written in the form

$$\begin{aligned} &e^x\{{}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{4}) - \frac{x}{2\alpha+1}{}_0F_1(-;\alpha+\frac{3}{2};\frac{x^2}{4})\} \\ &= {}_1F_1(\alpha;2\alpha+1;2x). \end{aligned} \tag{2.1}$$

Now, let

(2.2)
$$e^x\{{}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{4})-\frac{x}{2\alpha+1}{}_0F_1(-;\alpha+\frac{3}{2};\frac{x^2}{4})\}:=\sum_{n=0}^{\infty}a_nx^n.$$

Clearly, in the product of left-hand side of (2.2), it is not difficult to see that the coefficient a_n of x^n, after some simplification, is obtained as

$$\begin{aligned}a_n=&\frac{1}{n!}{}_2F_1\left[\begin{matrix}-\frac{n}{2},\frac{1-n}{2};\\ \alpha+\frac{1}{2};\end{matrix}\ 1\right]\\ &-\frac{1}{(2\alpha+1)}\frac{1}{(n-1)!}{}_2F_1\left[\begin{matrix}-\frac{n}{2}+\frac{1}{2},-\frac{n}{2}+1;\\ \alpha+\frac{3}{2};\end{matrix}\ 1\right].\end{aligned}\tag{2.3}$$

In both ${}_2F_1$, if we apply Gauss's summation theorem (1.7), then, after some simplification, we get

$$a_n=\frac{(\alpha)_n2^n}{n!(2\alpha+1)_n}.\tag{2.4}$$

Substituting the values of a_n in (2.2) and summing the series, we arrive at the right-hand side of (2.1). This completes the proof of (1.8).

In exactly the same manner, the result (1.9) can also be established.

Second Method : Use of contiguous relations.

It is just a simple exercise to prove the following contiguous relation between three ${}_1F_1$'s.

$$\begin{aligned}&{}_1F_1(\alpha;2\alpha+1;x)\\ &={}_1F_1(\alpha;2\alpha;x)-\frac{x}{2(2\alpha+1)}{}_1F_1(\alpha+1;2\alpha+2;x).\end{aligned}\tag{2.5}$$

Multiplying both sides of (2.5), by $e^{-x/2}$, we get

$$e^{-x/2}{}_1F_1(\alpha;2\alpha+1;x)$$

(2.6)

$$= e^{-x/2}{}_1F_1(\alpha;2\alpha;x) - \frac{x}{2(2\alpha+1)}e^{-x/2}{}_1F_1(\alpha+1;2\alpha+2;x).$$

But right-hand side can be evaluated with the help of Kummer's second theorem (1.4), and at once, we arrive at (1.8).

In exactly the same manner, the result (1.9) can be derived with the help of the following contiguous relation involving three ${}_1F_1$'s :

$${}_1F_1(\alpha;2\alpha-1;x)$$

(2.7)

$$= {}_1F_1(\alpha-1;2\alpha-2;x) + \frac{x}{2(2\alpha-1)}{}_1F_1(\alpha;2\alpha;x).$$

Third Method : Use of results (1.13) and (1.14) contiguous to the Preece's identity (1.12).

We start with the result (1.13), which can be written in the form

$$\begin{aligned}\{e^{-x/2}{}_1F_1(\alpha;2\alpha+1;x)\}^2 &= {}_1F_2(\alpha;\alpha+\frac{1}{2},2\alpha;\frac{x^2}{4})\\ &- \frac{x}{(2\alpha+1)}{}_1F_2(\alpha+1;\alpha+\frac{3}{2},2\alpha+1;\frac{x^2}{4})\\ &+ \frac{x^2}{4(2\alpha+1)^2}{}_1F_2(\alpha+1;\alpha+\frac{3}{2},2\alpha+2;\frac{x^2}{4}).\end{aligned} \tag{2.8}$$

On using the results (1.17) and (1.18) in the right-hand side of (2.8), we get

$$\begin{aligned}\{e^{-x/2}{}_1F_1(\alpha;2\alpha+1;x)\}^2 &= \left[{}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{16})\right]^2\\ &- \frac{x}{(2\alpha+1)}{}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{16}){}_0F_1(-;\alpha+\frac{3}{2};\frac{x^2}{16})\\ &+ \frac{x^2}{4(2\alpha+1)^2}\left[{}_0F_1(-;\alpha+\frac{3}{2};\frac{x^2}{16})\right]^2.\end{aligned} \tag{2.9}$$

Thus, we have

(2.10)
$$\{e^{-x/2}{}_1F_1(\alpha;2\alpha+1;x)\}^2 = \left\{{}_0F_1(-;\alpha+\frac{1}{2};\frac{x^2}{16}) - \frac{x}{2(2\alpha+1)}{}_0F_1(-;\alpha+\frac{3}{2};\frac{x^2}{16})\right\}^2$$

from which the result (1.8) follows immediately.

In exactly the same manner, the result (1.9) can also be established with the help of (1.14).

References

1. W. N. Bailey, *Products of generalized hypergeometric series*, Proc. London Math. Soc. (2) **28** (1928), 242–254.
2. W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, London (1935), and reprinted by Stechert-Hafner, New York, 1964.
3. E. E. Kummer, *Über die hypergeometrische Reihe ...*, J. Reine Angew. Math. **15** (1836), 39–83 and 127–172.
4. J. L. Lavoie, F. Grondin, and A. K. Rathie, *Generalizations of Watson's theorem on the sum of a* ${}_3F_2$, Indian J. Math. **34** (1992), 23–32.
5. C. T. Preece, *The product of two generalized hypergeometric functions*, Proc. London Math. Soc., (2) **22** (1924), 370–380.
6. E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
7. A. K. Rathie, *A short proof of Preece's identities and other contiguous results*, Rev. Mat. Estat. **15** (1997), 297–310.
8. A. K. Rathie and J. Choi, *A note on an identity due to Preece*, Far East J. Math. Sci. **6(2)** (1998), 205–209.
9. A. K. Rathie and J. Choi, *Another proof of Kummer's second theorem*, Comm. Korean Math. Soc. **13** (1998), 933–936.
10. A. K. Rathie and Y. Kim, *A generalization of Preece's identity*, Comm. Korean Math. Soc. **14** (1999), 217–222.
11. A. K. Rathie and V. Nagar, *On Kummer's second theorem involving product of generalized hypergeometric series*, Le Matematiche (Catania), **50** (1995), 35–38.