ANTI FUZZY CHARACTERISTIC IDEALS OF A BCK-ALGEBRA

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The concept of fuzzy sets was introduced by Zadeh [8]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, etc. Jun et al. [6] introduced the notion of fuzzy characteristic subalgebras/ideals of a BCK-algebra. They proved that a fuzzy ideal $\mu$ of a BCK-algebra is a fuzzy characteristic ideal if and only if each level ideal of $\mu$ is a characteristic ideal. S. M. Hong and Y. B. Jun [3] introduced the concept of fuzzy characteristic $\Gamma$-ideals of a $\Gamma$-ring, and they showed that a fuzzy characteristic $\Gamma$-ideal is characterized in terms of its level $\Gamma$-ideals. Recently, on the other hand, they also [2] defined the notions of anti fuzzy ideals of a BCK-algebra. The present author [5], modifying S. M. Hong and Y. B. Jun's idea, introduced anti fuzzy prime ideals of a commutative BCK-algebra, and proved that every anti fuzzy prime ideal of a commutative BCK-algebra is an anti fuzzy ideal.

In this paper, we define the notion of anti fuzzy characteristic ideals of BCK-algebras, and obtain some results about it.

We begin with several preliminaries definitions and propositions.

**Definition 1.** An algebra $(X, *, 0)$ of type $(2,0)$ is called a BCK-algebra if it satisfies the following axioms: for all $x, y, z \in X$,

(a) $((x * y) * (x * z)) * (z * y) = 0$,
(b) $(x * (x * y)) * y = 0$,
(c) $x * x = 0$,
(d) $0 * x = 0$,
(e) $x * y = 0$ and $y * x = 0$ imply $x = y$.

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A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $x * y = 0$. This ordering is called BCK-ordering.

**Proposition 1.** In any BCK-algebra $X$, the following hold: for all $x, y, z \in X$,

1. $x * 0 = x$,
2. $(x * y) * z = (x * z) * y$,
3. $x * y \leq x$,
4. $(x * y) * z \leq (x * z) * (y * z)$,
5. $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

**Definition 2.** [4] A non-empty subset $I$ of a BCK-algebra $X$ is called an ideal of $X$ if

1. $0 \in I$,
2. $x * y \in I$ and $y \in I$ imply $x \in I$.

**Definition 3.** [8] Let $S$ be a non-empty set. A fuzzy subset $\mu$ of $S$ is a function $\mu : S \rightarrow [0, 1]$.

**Definition 4.** [1] Let $\mu$ be a fuzzy subset of $S$. Then for $t \in [0, 1]$, the level subset of $\mu$ is the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$.

**Definition 5.** [7] Let $X$ be a BCK-algebra. A fuzzy subset $\mu$ of $X$ is called a fuzzy subalgebra of $X$ if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

**Definition 6.** [7] Let $X$ be a BCK-algebra. A fuzzy subset $\mu$ of $X$ is called a fuzzy ideal of $X$ if, for $x, y \in X$,

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

**Definition 7.** [2] A fuzzy subset $\mu$ of a BCK-algebra $X$ is called an anti fuzzy subalgebra of $X$ if

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$$

for all $x, y \in X$. 
Proposition 2. [2] Let \( \mu \) be an anti fuzzy subalgebra of a BCK-algebra \( X \). Then \( \mu(0) \leq \mu(x) \) for every \( x \in X \).

Definition 8. [2] A fuzzy subset \( \mu \) of a BCK-algebra \( X \) is called an anti fuzzy ideal of \( X \) if

1. \( \mu(0) \leq \mu(x) \),
2. \( \mu(x) \leq \max\{\mu(x \ast y), \mu(y)\} \),

for all \( x, y \in X \).

Clearly, every anti fuzzy ideal \( \mu \) of a BCK-algebra \( X \) is an anti fuzzy subalgebra of \( X \), but not conversely.

Definition 9. If \( \mu \) is a fuzzy subset of \( X \) and \( \alpha \) is a function from \( X \) into itself, we define a function \( \mu^\alpha \) from \( X \) into \([0, 1]\) by \( \mu^\alpha(x) = \mu(\alpha(x)) \) for every \( x \in X \).

Suppose that \( \mu \) is an anti fuzzy subalgebra of a BCK-algebra \( X \) and \( \alpha \) is an endomorphism of \( X \). Then

\[
\mu^\alpha(x \ast y) = \mu(\alpha(x \ast y)) \\
= \mu(\alpha(x) \ast \alpha(y)) \\
\leq \max\{\mu(\alpha(x)), \mu(\alpha(y))\} \\
= \max\{\mu^\alpha(x), \mu^\alpha(y)\},
\]

for all \( x, y \in X \) and

\[
\mu^\alpha(x) = \mu(\alpha(x)) \\
= \max\{\mu(\alpha(x)), \mu(\alpha(x))\} \\
\geq \mu(\alpha(x) \ast \alpha(x)) \\
= \mu(\alpha(x \ast x)) \\
= \mu^\alpha(0),
\]

for every \( x \in X \). Hence we have the following proposition.

Proposition 3. Let \( \mu \) be an anti fuzzy subalgebra of a BCK-algebra \( X \) and let \( \alpha \) be an endomorphism of \( X \). Then

1. \( \mu^\alpha \) is an anti fuzzy subalgebra of \( X \),
2. \( \mu^\alpha(0) \leq \mu^\alpha(x) \) for every \( x \in X \).
PROPOSITION 4. Let \( \mu \) be an anti fuzzy ideal of \( X \) and let \( \alpha \) be an endomorphism of \( X \). Then the following holds for all \( x, y, z \in X \),

1. if \( x \leq y \), then \( \mu^\alpha(x) \leq \mu^\alpha(y) \).
2. \( \mu^\alpha(x * y) \leq \max\{\mu^\alpha(x * z), \mu^\alpha(z * y)\} \).
3. if \( \mu^\alpha(x * y) = \mu^\alpha(0) \), then \( \mu^\alpha(x) \leq \mu^\alpha(y) \).
4. \( \max\{\mu^\alpha(x * y), \mu^\alpha(y)\} = \max\{\mu^\alpha(x), \mu^\alpha(y)\} \).
5. if \( X \) is bounded, then \( \max\{\mu^\alpha(x), \mu^\alpha(1 * x)\} = \mu^\alpha(1) \).
6. if \( x \leq y \), then \( \mu^\alpha(y) = \max\{\mu^\alpha(y * x), \mu^\alpha(x)\} \).

Proof. (1) If \( x \leq y \), then we have \( x * y = 0 \). Thus,

\[
\mu^\alpha(x) = \mu(\alpha(x)) \\
\leq \max\{\mu(\alpha(x) * \alpha(y)), \mu(\alpha(y))\} \\
= \max\{\mu(0), \mu(\alpha(y))\} \\
= \mu(\alpha(y)) = \mu^\alpha(y).
\]

(2) From (a) of definition of BCK-algebra and (1), we have that \( \mu^\alpha((x * y) * (x * z)) \leq \mu^\alpha(z * y) \). Thus,

\[
\mu^\alpha(x * y) = \mu(\alpha(x * y)) \\
= \mu(\alpha(x) * \alpha(y)) \\
\leq \max\{\mu((\alpha(x) * \alpha(y)) * (\alpha(x) * \alpha(z))), \mu(\alpha(x) * \alpha(z))\} \\
= \max\{\mu^\alpha((x * y) * (x * z)), \mu^\alpha(x * z)\} \\
\leq \max\{\mu^\alpha(z * y), \mu^\alpha(x * z)\}.
\]

(3) Suppose that \( \mu^\alpha(x * y) = \mu^\alpha(0) \). Then

\[
\mu^\alpha(x) = \mu(\alpha(x)) \\
\leq \max\{\mu(\alpha(x) * \alpha(y)), \mu(\alpha(y))\} \\
= \max\{\mu(\alpha(x * y)), \mu(\alpha(y))\} \\
= \max\{\mu^\alpha(x * y), \mu^\alpha(y)\} \\
= \max\{\mu^\alpha(0), \mu^\alpha(y)\} \\
= \max\{\mu(\alpha(0)), \mu(\alpha(y))\} \\
= \max\{\mu(0), \mu(\alpha(y))\} \\
= \mu(\alpha(y)) \\
= \mu^\alpha(y).
\]
(4) Since \( x \ast y \leq x \), we have \( \mu^\alpha(x \ast y) \leq \mu^\alpha(x) \) by (1). On the other hand,

\[
\mu^\alpha(x) = \mu(\alpha(x)) \\
\leq \max\{\mu(\alpha(x) \ast \alpha(y)), \mu(\alpha(y))\} \\
= \max\{\mu(\alpha(x \ast y)), \mu(\alpha(y))\} \\
= \max\{\mu^\alpha(x \ast y), \mu^\alpha(y)\}.
\]

It follows that \( \max\{\mu^\alpha(x \ast y), \mu^\alpha(y)\} = \max\{\mu^\alpha(x), \mu^\alpha(y)\} \).

(5) If \( X \) is bounded, then by (1), \( \mu^\alpha(1) \geq \max\{\mu^\alpha(x), \mu^\alpha(1 \ast x)\} \).

On the other hand,

\[
\mu^\alpha(1) = \mu(\alpha(1)) \\
\leq \max\{\mu(\alpha(1) \ast \alpha(x)), \mu(\alpha(x))\} \\
= \max\{\mu(\alpha(1 \ast x)), \mu(\alpha(x))\} \\
= \max\{\mu^\alpha(1 \ast x), \mu^\alpha(x)\}.
\]

Hence (5) holds.

(6) is obtained from (1) and (4).

**Proposition 5.** Let \( \mu \) be an anti fuzzy ideal of \( X \) and let \( \alpha : X \rightarrow X \) be an onto homomorphism. Then \( \mu^\alpha \) is an anti fuzzy ideal of \( X \).

**Proof.** For all \( x \in X \), we have that

\[
\mu^\alpha(x) = \mu(\alpha(x)) \geq \mu(0) = \mu(\alpha(0)) = \mu^\alpha(0).
\]

Next for any \( x, y \in X \),

\[
\mu^\alpha(x) = \mu(\alpha(x)) \leq \max\{\mu(\alpha(x) \ast y), \mu(y)\}.
\]

Since \( \alpha \) is onto, there is \( z \in X \) such that \( \alpha(z) = y \). It follows that

\[
\mu^\alpha(x) \leq \max\{\mu(\alpha(x) \ast z), \mu(z)\} \\
= \max\{\mu(\alpha(x) \ast \alpha(z)), \mu(\alpha(z))\} \\
= \max\{\mu(\alpha(x \ast z)), \mu(\alpha(z))\} \\
= \max\{\mu^\alpha(x \ast z), \mu^\alpha(z)\}.
\]

Since \( y \) is an arbitrary element of \( X \), the above result is true for all \( z \in X \), i.e., \( \mu^\alpha(x) \leq \max\{\mu^\alpha(x \ast z), \mu^\alpha(z)\} \) for all \( x, z \in X \). Thus \( \mu^\alpha \) is an anti fuzzy ideal of \( X \).
Definition 10. An anti fuzzy subalgebra (ideal) \( \mu \) of \( X \) is called an anti fuzzy characteristic subalgebra (ideal) of \( X \) if \( \mu(\alpha(x)) = \mu(x) \) for all \( x \in X \) and all \( \alpha \in \text{Aut}(X) \).

Definition 11. [2] Let \( \mu \) be a fuzzy subset of a BCK-algebra \( X \). Then for \( t \in [0,1] \), the set

\[
\mu^t := \{ x \in X \mid \mu(x) \leq t \}
\]

is called the lower \( t \)-level cut of \( \mu \).

Proposition 6. [2] Let \( \mu \) be a fuzzy subset of a BCK-algebra \( X \). Then it is an anti fuzzy ideal of \( X \) if and only if for every \( t \in [0,1] \), \( t \geq \mu(0) \), the lower \( t \)-level cut \( \mu^t \) is an ideal of \( X \).

Proposition 7. Let \( \mu \) be an anti fuzzy characteristic subalgebra of a BCK-algebra \( X \). Then each lower \( t \)-level cut of \( \mu \) is a characteristic subalgebra of \( X \).

Proof. Let \( t \in \text{Im}(\mu) \), \( \alpha \in \text{Aut}(X) \) and \( x \in \mu^t \). Since \( \mu \) is an anti fuzzy characteristic subalgebra of \( X \), we have \( \mu(\alpha(x)) = \mu(x) \leq t \). It follows that \( \alpha(x) \in \mu^t \) and hence \( \alpha(\mu^t) \subseteq \mu^t \). To show the reverse inclusion, let \( x \in \mu^t \) and let \( y \in X \) be such that \( \alpha(y) = x \). Then \( \mu(y) = \mu(\alpha(y)) = \mu(x) \leq t \), so \( y \in \mu^t \). It follows that \( x = \alpha(y) \in \alpha(\mu^t) \). Hence \( \mu^t \subseteq \alpha(\mu^t) \). Thus \( \mu^t \) is a characteristic subalgebra of \( X \), for each \( t \in \text{Im}(\mu) \).

The proof of the following lemma is obvious, and we omit the proof.

Lemma 1. Let \( \mu \) be an anti fuzzy subalgebra (ideal) of \( X \) and let \( x \in X \). Then \( \mu(x) = t \) if and only if \( x \in \mu^t \) and \( x \not\in \mu^s \) for all \( s < t \).

Now we consider the converse of Proposition 7.

Proposition 8. Let \( \mu \) be an anti fuzzy subalgebra of \( X \). If each lower \( t \)-level cut \( \mu^t \) is a characteristic subalgebra of \( X \), then \( \mu \) is an anti fuzzy characteristic subalgebra of \( X \).

Proof. Let \( x \in X \), \( \alpha \in \text{Aut}(X) \) and \( \mu(x) = t \). Then \( x \in \mu^t \) and \( x \not\in \mu^s \) for all \( s < t \), by Lemma 1. Since \( \alpha(\mu^t) = \mu^t \) by hypothesis, we have \( \alpha(x) \in \mu^t \). Hence \( \mu(\alpha(x)) \leq t \). Let \( s = \mu(\alpha(x)) \). We now show that \( s = t \). Indeed, suppose that \( s < t \). Then \( \alpha(x) \in \mu^s = \alpha(\mu^s) \).
Since $\alpha$ is one-to-one, we have $x \in \mu^\alpha$. This is a contradiction. Thus $\mu(\alpha(x)) = t = \mu(x)$. It follows that $\mu$ is an anti fuzzy characteristic subalgebra of $X$.

The proofs of the following propositions are similar to those of Propositions 7 and 8.

**Proposition 9.** If $\mu$ is an anti fuzzy characteristic subalgebra of $X$, then each lower $t$-level cut of $\mu$ is a characteristic ideal of $X$.

**Proposition 10.** Let $\mu$ be an anti fuzzy ideal of $X$. If each lower $t$-level cut of $\mu$ is a characteristic ideal of $X$, then $\mu$ is an anti fuzzy characteristic ideal of $X$.

**References**


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