A NOTE ON SAALSCHÜTZ'S THEOREM

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1. Introduction and Preliminaries

The generalized hypergeometric function with p numerator and q denominator parameter is defined by

\[ \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \]

where the Pochhammer symbol \((\alpha)_n\) (or the shifted factorial, since \((1)_n = n!\)), is defined by, \(\alpha\) any complex number,

\[ (\alpha)_n := \begin{cases} \alpha(\alpha+1) \cdots (\alpha+n-1) & \text{if } n \in \mathbb{N} := \{1, 2, 3, \cdots \} \\ 1 & \text{if } n = 0, \end{cases} \]

which, in view of the fundamental functional relation of the Gamma function \(\Gamma\)

\[ \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \text{and} \quad \Gamma(1) = 1, \]

is written in the equivalent form:

\[ (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \]

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where $\Gamma$ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$\{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right\},$$

where $\gamma$ is the Euler-Mascheroni's constant defined by

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \approx 0.577215664\ldots.$$

From the definition (1.2), it is easy to see that

$$\binom{\alpha}{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k} \quad (0 \leq k \leq n),$$

which, for $\alpha = 1$, reduces immediately to

$$\binom{n-k}{k} = \frac{(-1)^kn!}{(-n)_k} \quad (0 \leq k \leq n; \ n \in \mathbb{N}).$$

The generalized binomial coefficient $\binom{\alpha}{n}$ is defined by

$$\binom{\alpha}{n} := \left\{ \begin{array}{ll} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} & \text{if } \ n \in \mathbb{N} \\ 1 & \text{if } \ n = 0, \end{array} \right.$$ 

which, in virtue of (1.2) and (1.3), is expressed in the following equivalent form:

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} = \frac{(-1)^n(-\alpha)_n}{n!},$$

from which (1.2) may be extended to

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)(1-\alpha)_n} = \frac{(-1)^n}{(1-\alpha)_n} \quad (\alpha \neq 0, \pm 1, \pm 2, \cdots; \ n \in \mathbb{N}).$$

In this note we first recall a generalized form of the Saalschütz's theorem which is one of the important and useful theorems in $\,_{3}F_{2}$. We also investigate connections among some theorems associated with the generalized hypergeometric series.
2. Relations among Theorems

We start with a transformation formula due to Whipple [5, p. 263]

\[
4F3 \begin{bmatrix} t, x, y, z; \\ u, v, w, 1 \end{bmatrix} = \frac{\Gamma(v + w - t)\Gamma(1 + x - u)\Gamma(1 + y - u)\Gamma(1 + z - u)}{\Gamma(1 + y + z - u)\Gamma(1 + z + x - u)\Gamma(1 + x + y - u)\Gamma(1 - u)} \times 7F6 \begin{bmatrix} a, 1 + \frac{1}{2}a, w - t, v - t, x, y, z; \\ \frac{1}{2}a, v, w, 1 + y + z - u, 1 + z + x - u, 1 + x + y - u; 1 \end{bmatrix},
\]

which transforms a terminating well-poised \(7F6\) into a series \(4F3\) and \(a = x + y + z - u, u + v + w = t + x + y + z + 1,\) while one of the four \(t, x, y, z\) is a negative integer.

Setting \(v = t\) in (2.1) reduces immediately to Saalschütz's theorem:

\[
3F2 \begin{bmatrix} x, y, z; \\ u, w; 1 \end{bmatrix} = \frac{\Gamma(u)\Gamma(1 - w + x)\Gamma(1 - w + y)\Gamma(1 - w + z)}{\Gamma(1 - w)\Gamma(u - x)\Gamma(u - y)\Gamma(u - z)}
\]

provided that \(x, y\) or \(z\) is a negative integer and \(u + w = x + y + z + 1.\)

By using (1.4), (2.2) is written in the following form

\[
3F2 \begin{bmatrix} -n, a, b; \\ c, 1 - c + a + b - n; 1 \end{bmatrix} = \frac{(c - a)n(c - b)n}{(c)n(c - a - b)n} (n \in \mathbb{N} \cup \{0\}).
\]

A useful well-known asymptotic formula for Gamma function is also provided:

\[
\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta} \left[ 1 + O \left( \frac{1}{z} \right) \right] \quad (z \to \infty; \ |\arg z| < \pi).
\]

Taking the limit in (2.3) as \(n \to \infty\) with the aid of (2.4) yields Gauss's summation formula

\[
2F1 (a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \quad (c \neq 0, -1, -2, \ldots; \ \text{Re}(c - a - b) > 0),
\]
which plays a vital role in the theory of hypergeometric series and can be deduced in several ways.

Replacing $a$ or $b$ by a nonpositive integer $-n$, we have a summation formula

\[
(2.6) \quad 2F_1(-n, b; c; 1) = \frac{(c - b)_n}{(c)_n} \quad (n \in \mathbb{N} \cup \{0\}; \ c \neq 0, -1, -2, \ldots),
\]

which incidentally can be shown to be equivalent to so-called Vandermonde’s convolution theorem (see Choi [2, p. 159]; also Srivastava et al. [4, p. 19]):

\[
(2.7) \quad \sum_{k=0}^{n} \binom{\lambda}{k} \binom{\mu}{n-k} = \binom{\lambda + \mu}{n} \quad (n \in \mathbb{N} \cup \{0\}).
\]

Indeed, using (1.8) and (1.10), the left side of (2.7), say $S_n$, becomes

\[
S_n = \sum_{k=0}^{n} \frac{(-1)^k(-\lambda)_k}{k!} \left[ \frac{(-1)^{n-k}(-\mu)_{n-k}}{(n-k)!} \right] = \sum_{k=0}^{n} \frac{(-1)^k(-\lambda)_k}{k!} \cdot \frac{(-1)^n(-1)^n(-\mu)_n}{(1+\mu-n)_k \cdot (-1)^k n!}
\]

\[
= \frac{(-1)^n(-\mu)_n}{n!} \sum_{k=0}^{n} \frac{(-\lambda)_k}{(1+\mu-n)_k k!}
\]

\[
= \frac{(-1)^n(-\mu)_n}{n!} \cdot 2F_1(-n, -\lambda; 1+\mu-n; 1),
\]

which, for $b = -\lambda$ and $c = 1 + \mu - n$, in view of (2.6), yields

\[
S_n = \frac{(-1)^n(-\lambda - \mu)_n}{n!}.
\]

This, in virtue of (1.10), completes the proof of (2.7).

Recall a transformation formula for $2F_1$:

\[
(2.8) \quad 2F_1(c-a, c-b; c; z) = (1-z)^{a+b-c} \cdot 2F_1(a, b; c; z),
\]
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which is due to Euler. Rainville [4, pp. 86-88] obtained (2.3) by interpreting (2.8) as an identity involving three power series. Conversely, as suggested in Bailey [1, p. 245], we can prove (2.8) by making use of (2.3). Indeed, let

\[(1 - z)^{a+b-c} F_1(a, b; c; z) := \sum_{n=0}^{\infty} a_n z^n.\]

Starting with the left side of (2.9), we have

\[(1 - z)^{a+b-c} F_1(a, b; c; z) = \left[ \sum_{n=0}^{\infty} \frac{(c-a-b)_n}{n!} z^n \right] \left[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \right] = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \frac{(c-a-b)_n (a)_n (b)_n}{(n-k)! (c)_k k!} \right] z^n,\]

which, in view of (1.7) and (1.8) with equating the coefficients of $z^n$, yields

\[a_n = \frac{(c-a-b)_n}{n!} \begin{bmatrix} -n, & a, & b; \\ c, 1 - c + a + b - n; & 1 \end{bmatrix},\]

which, using (2.3), becomes

\[a_n = \frac{(c-a)_n (c-b)_n}{n! (c)_n}.\]

Thus completes the proof of (2.8).

Choi [2, p. 160] proved the following formula by mathematical induction method:

\[(2.10) \quad \frac{(A)_n (B)_n}{(C)_n} = \sum_{k=0}^{n} \binom{n}{k} \frac{(C-B)_k (C-A)_k}{(C)_k} (A + B - C)_{n-k},\]

which can incidentally be seen to be equivalent to Saalschütz's theorem (2.3). Indeed, applying (1.7) and (1.9) to (2.10), we obtain

\[(2.11) \quad \begin{bmatrix} -n, & C - A, & C - B; \\ C, 1 + C - A - B - n; & 1 \end{bmatrix} = \frac{(A)_n (B)_n}{(A + B - C)_n (C)_n},\]
which, for $C = c$, $C - A = a$ and $C - B = b$, immediately yields (2.3) and vice versa.

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References


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