THE RADIUS OF CONVEXITY FOR THE CLASS $K^{(2)}$

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1. Introduction

Let $S$ denote the class of functions $f$ of a complex variable $z$, analytic and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$, and normalized by $f(0) = f'(0) - 1 = 0$ and hence with the Taylor expansion

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad z \in \Delta.$$  

Let $K$ denote the subclass of $S$ consisting of functions $f$ for which $f(\Delta)$ is a convex set. Furthermore, let $S^{(2)}$ denote the class of odd functions in $S$, i.e., the functions with the expansion

$$g(z) = z + c_3 z^3 + c_5 z^5 + \cdots + c_{2n+1} z^{2n+1} + \cdots, \quad z \in \Delta.$$  

For each function $f \in S$, the square root transform

$$g(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots$$

is an odd univalent function. Conversely, it is easy to see that every odd function $g \in S$ is the square-root transform of some $f \in S$. We define $K^{(2)}$ be the class of functions which are square-root transforms of functions in $K$.

The one of the geometric properties for the class $S$ is that every $f(z)$ in $S$ is not convex. Near the origin each function $f \in S$ is close to the identity mapping. It is to be expected that $f$ will map small circles $|z| = \rho$ onto curves which bound convex domains.

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Theorem 1.1. [1] For every positive number $\rho \leq 2 - \sqrt{3}$, each function $f \in S$ maps the disc $|z| < \rho$ onto a convex domain. This is false for every $\rho > 2 - \sqrt{3}$.

This number $\rho = 2 - \sqrt{3} = 0.267\ldots$ is called the radius of convexity for the class $S$. Let $h(z) = z(1-z)^{-1} \in K$. Then we have $\sqrt{h(z^2)} \notin K$, i.e., $K^{(2)}$ is not the subclass of $K$. Thus we would find the radius of convexity for the class $K^{(2)}$.

2. Preliminaries

Theorem 2.1. ([1], Growth and Distortion theorem) If $f \in S$ and $|z| = r < 1$ then

$$\frac{r}{(1 + r)^2} \leq |f(z)| \leq \frac{r}{(1 - r)^2}$$

and

$$\frac{1 - r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^3}.$$  

For each $z \in \Delta$, $z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the Koebe function.

Theorem 2.2. [1] For each $f \in S$,

$$\frac{1 - r}{1 + r} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + r}{1 - r}, \quad |z| = r < 1.$$  

For each $z \in \Delta, z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the Koebe function.

Theorem 2.3. For odd functions $h \in S^{(2)}$

$$\frac{r}{1 + r^2} \leq |h(z)| \leq \frac{r}{1 - r^2}$$

and

$$\frac{1 - r^2}{(1 + r^2)^2} \leq |h'(z)| \leq \frac{1 + r^2}{(1 - r^2)^2}, \quad |z| = r < 1.$$  

Proof. Let $h(z) = \sqrt{f(z^2)}$ for some $f \in S$, then

$$\sqrt{\frac{r^2}{(1 + r^2)^2}} \leq |h(z)| \leq \sqrt{\frac{r^2}{(1 - r^2)^2}}.$$
The radius of convexity for the class $K^{(2)}$.

Thus

$$\frac{r}{1 + r^2} \leq |h(z)| \leq \frac{r}{1 - r^2}, \quad |z| = r < 1.$$ 

Since

$$\frac{1 - r}{1 + r} \leq \frac{|z f'(z)|}{f(z)} \leq \frac{1 + r}{1 - r}$$

and

$$\frac{zh'(z)}{h(z)} = \frac{z^2 f'(z^2)}{f(z^2)},$$

$$\frac{1 - r^2}{1 + r^2} \leq \frac{|zh'(z)|}{h(z)} \leq \frac{1 + r^2}{1 - r^2}$$

and

$$|h'(z)| = \left| \frac{z f'(z^2) h(z)}{f(z^2)} \right|, \quad |z| = r < 1.$$ 

Thus

$$\frac{1 - r^2}{(1 + r^2)^2} \leq |h'(z)| \leq \frac{1 + r^2}{(1 - r^2)^2}, \quad |z| = r < 1.$$ 

3. Main Results

**Lemma 3.1.** For each $f \in K$,

$$\frac{1}{(1 + r)^2} \leq |f'(z)| \leq \frac{1}{(1 - r)^2}, \quad |z| = r < 1.$$ 

For each $z \in \Delta$, $z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the function $l(z) = z(1 - z)^{-1}$.

**Lemma 3.2.** For convex function $f \in K$,

$$\frac{r}{1 + r} \leq |f(z)| \leq \frac{r}{1 - r}, \quad |z| = r < 1,$$

with equality occurring only for functions of the form

$$f(z) = \frac{z}{1 - e^{i\varphi} z}, \quad 0 \leq \varphi \leq 2\pi.$$ 

The growth of $K^{(2)}$ would be obtained by the following theorem.
Theorem 3.3. For $h \in K^{(2)}$, 
\[
\frac{r}{\sqrt{1 + r^2}} \leq |h(z)| \leq \frac{r}{\sqrt{1 - r^2}}, \quad |z| = r < 1.
\]

Proof. Let $h(z) = \sqrt{f(z^2)}$ and $f \in K$. Then by Lemma 3.2,
\[
|h(z)| = |\sqrt{f(z^2)}| \leq \sqrt{\frac{r^2}{1 - r^2}} = \frac{r}{\sqrt{1 - r^2}}
\]
and
\[
\frac{r}{\sqrt{1 + r^2}} \leq |h(z)|, \quad |z| = r < 1.
\]
If $h \in K^{(2)}$, then we have
\[
\frac{r}{1 + r} \leq \frac{r}{\sqrt{1 + r^2}} \leq |h(z)| \leq \frac{r}{\sqrt{1 - r^2}} \leq \frac{r}{1 - r}, \quad |z| = r < 1
\]
But $K^{(2)}$ is not the subclass of convex functions.

Lemma 3.4. For each $f \in K$, 
\[
\frac{1}{1 + r} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1}{1 - r}, \quad |z| = r < 1.
\]
For each $z \in \Delta$, $z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the function $l(z) = z/(1 - z)$.

Lemma 3.5. For each $f \in K$, 
\[
-\frac{2r}{1 + r} \leq \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} \leq \frac{2r}{1 - r}, \quad |z| = r < 1.
\]

Theorem 3.6. For every positive number $\sigma \leq \sqrt{5 - \sqrt{17}/2}$, each function $h \in K^{(2)}$ maps the disk $\Delta_\sigma = \{z : |z| < \sigma\}$ onto a convex domain and $\sqrt{5 - \sqrt{17}/2} > 2 - \sqrt{3}$

Proof. For each $f \in K$ and $h = \sqrt{f(z^2)} \in K^{(2)}$, 
\[
\Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} = \Re \left\{ 2 + \frac{2zf''(z^2)}{f'(z^2)} - \frac{z^2f'(z^2)}{f(z^2)} \right\}
\]
and
\[ \text{Re} \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > 0, \quad |z| = r < \frac{\sqrt{5} - \sqrt{17}}{2} \]

by Lemma 3.4 and 3.5. Thus \( h \) maps such a disk \( \{ z : |z| < \sqrt{5 - \sqrt{17}} / 2 \} \) onto a convex domain.

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References


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