East Asian Math J 15(1999), No. 2, pp 153-163

ON p_n-SEQUENCES OF UNIVERSAL ALGEBRAS

JUNG-RAE CHO

ABSTRACT We study how the p_n-sequence of a universal algebra determine the structure of the algebra Regarding term equivalent algebras as the same algebras, we consider the problem when the algebras are groupoids.

1. Introduction

A term $f(x_1, x_2, \ldots, x_n)$ over an abstract algebra $\mathcal{A} = (A, \Omega)$ is called *n-ary* if it involves n distinct variables and *essential* if it depends on each variable it involves in the sense that, for each $i = 1, 2, \ldots, n$, there are $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ and b, c in A such that

$$f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n).$$

We denote by $p_n(\mathcal{A})$ the number of essentially n-ary term functions over $\mathcal{A}$, and the sequence $(p_0(\mathcal{A}), p_1(\mathcal{A}), p_2(\mathcal{A}), \ldots)$ is called the *p_n-sequence* of $\mathcal{A}$

A groupoid is called *trivial* if it has only one element and *proper* if the term xy is essentially binary.

Two algebras (A, Ω_1) and (A, Ω_2) on the same underlying set A are said to be *term equivalent* if they have the same term functions, that is, any Ω_1-term can be written as an Ω_2-term and vice versa.

Received February 10, 1999 Revised May 12, 1999

1991 Mathematics Subject Classification 08A40, 20N02

Key words and phrases p_n-sequences, clone, essentially n-ary

The present research was supported by Pusan National University Research Grant, 1995

For simplicity of our notation, we inductively define groupoid terms by $xy^1 = xy$ and $xy^{k+1} = (xy^k)y$, and use the expression $x_1x_2 \cdots x_{n-1}x_n$ for $((\cdots(x_1x_2)\cdots)x_{n-1})x_n$

A groupoid $(G, \cdot)$ is said to be *medial* if it satisfies the identity $(xy)(uv) = (xu)(yv)$, and *distributive* if it satisfies the $x(yz) = (xy)(xz)$ and $(xy)z = (xz)(yz)$. A commutative idempotent groupoid is called a *semilattice* if it is a semigroup, a *near-semilattice* if it satisfies $xy^2 = xy$, and a *Steiner quasigroup* if it satisfies $xy^2 = x$.

An *affine space over a field* K is algebraically defined to be the full idempotent reduct of a vector space over K ([2,13,15,17]). However, when the base field is the Galois field $GF(3)$ with three element, any affine space over $GF(3)$ is term equivalent to a medial Steiner quasigroup ([12]). Thus we will treat an affine space over $GF(3)$ as a medial Steiner quasigroup in this paper.

We say a sequence $\mathbf{a} = (a_0, a_1, a_2, \dots)$ (finite or infinite) of cardinals is called *representable* if there is an algebra A such that $\mathbf{p}(A) = \mathbf{a}$, that is, $p_n(A) = a_n$ for all n, and call $\mathbf{a}$ the p_n-sequence of A in this case. If, furthermore, A is from a given class K of algebra, we say that $\mathbf{a}$ is *representable in* K or A *represents* $\mathbf{a}$ in K.

A *clone* on a set A is a collection of operations on A which is closed under compositions and contains all projections. A clone C is called *minimal* if the lattice of subclones of C has only two elements. This means that $\mathrm{Card}(A) \neq 1$ and any term in C together with projections generates C

For further concepts and notations not defined in this paper, we refer the readers to [10] and [11].

A term $f(x_1, x_2, \dots, x_n)$ over a groupoid $(G, \cdot)$ will be called *linear term* if each variable appears at most once in the expression.

2. Theorems and proofs

THEOREM 1. *Let $(G, \cdot)$ be a nontrivial Steiner quasigroup. Then the following conditions are equivalent:*

(1) *$(G, \cdot)$ is an affine space over $GF(3)$;*
(2) *$(G, \cdot)$ is medial;*

(3) *The clone of* $(G,\cdot)$ *is minimal;*

(4) *The term* $f(x_1,x_2,x_3,x_4,x_5) = ((x_1x_2)(x_3x_4))x_5$ *is symmetric;*

(5) *For a certain* $n \geq 4$, *an* n*-ary term admits a nontrivial permutation.*

PROOF. The fact that the condition (i) implies any of the remaining one is not hard to check except the implication (i) $\Rightarrow$ (iii), which can be deduced from [14]. The implication (ii) $\Rightarrow$ (i) is contained in [12]. Using [8, Lemma 3.2], one can easily prove (iii) $\Rightarrow$ (ii). Now we prove (iv) $\Rightarrow$ (i) If $f(x_1,\cdots,x_5)$ is symmetric, then we obtain that

$$\begin{aligned}(x_1x_2)(x_3x_4) &= ((x_1x_2)(x_3x_4))((x_1x_2)(x_3x_4))\\ &= ((x_1x_3)(x_2x_4))((x_1x_2)(x_3x_4))\\ &= ((x_1x_2)(x_3x_4))((x_1x_3)(x_2x_4))\\ &= ((x_1x_3)(x_2x_4))((x_1x_3)(x_2x_4))\\ &= (x_1x_3)(x_2x_4).\end{aligned}$$

The implication (v) $\Rightarrow$ (ii) follows from [4,Theorem 4].

Now we recall a theorem of Gätzer and Padmanabhan from [12].

PROPOSITION 2. *If* A *is an algebra, then* A *is a nontrivial affine space over* $GF(3)$ *if and only if* $p_n(A) = \frac{2^n-(-1)^n}{3}$ *for all* n. *Moreover, if* A *is a groupoid, then it suffices to assume* $p_n(A) = \frac{2^n-(-1)^n}{3}$ *only for* $n = 0,1,2,3,4$.

With this theorem is connected

PROPOSITION 3. *([6,Theorem]) Let* $(G,\cdot)$ *be an idempotent groupoid. Then* $(G,\cdot)$ *is a nontrivial affine space over* $GF(3)$ *if and only if* $p_4(G,\cdot) = 5$.

Note that there exist idempotent groupoids $(G,\cdot)$ satisfying $p_n(A) = \frac{2^n-(-1)^n}{3}$ for all $n \leq 3$ which are not affine spaces over $GF(3)$ ([16]) This means that $p_4(G,\cdot) = 5$ is the first number of the p_n-sequence which uniquely determines the structure of an idempotent groupoid, and such groupoids are affine spaces over $GF(3)$ (see Theorem 9)

THEOREM 4. *Let $(G,\cdot)$ be a commutative idempotent groupoid. Then $(G,\cdot)$ is a nontrivial affine space over $GF(3)$ if and only if $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ for some $n \geq 4$.*

PROOF. If $(G,\cdot)$ is a nontrivial affine space over $GF(3)$, i.e., $(G,\cdot) = (G, 2x+2y)$, where $(G,+)$ is an abelian group of exponent 3, then $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ for all n by [2]. Let now $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ for some $n \geq 4$. Then $(G,\cdot)$ is not a semilattice since $p_n(G,\cdot) = 1$ for all n if $(G,\cdot)$ is a semilattice. If $(G,\cdot)$ is also not an affine space over $GF(3)$, then by [7,Theorem 1] we obtain that $p_n(G,\cdot) \geq 3^{n-1}$ for all $n \geq 4$. Hence, $\frac{2^n-(-1)^n}{3} \geq 3^n$ for all $n \geq 4$, which is not true. Thus $(G,\cdot)$ is an affine space over $GF(3)$.

In this connection we conjecture that if $(G,\cdot)$ is an idempotent groupoid (not necessarily commutative), then $(G,\cdot)$ is a nontrivial affine space over $GF(3)$ if and only if $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ for some $n \geq 4$ (compare with Theorem 6).

THEOREM 5. *Let $(G,\cdot)$ be a commutative idempotent groupoid. Then $(G,\cdot)$ is a nontrivial affine space over $GF(3)$ if and only if $p_3(G,\cdot) = 3$ and the clone of $(G,\cdot)$ is minimal.*

PROOF. If $(G,\cdot)$ is a nontrivial affine space over $GF(3)$, then trivially $p_3(G,\cdot) = 3$ and the clone of $(G,\cdot)$ is minimal by [14] Assume that $p_3(G,\cdot) = 3$ and the clone of $(G,\cdot)$ is minimal. By [9,Theorem 1.2], $(G,\cdot)$ is a nontrivial distributive Steiner quasigroup. Then by Theorem 1 $(G,\cdot)$ is an affine space over $GF(3)$.

Note that in this theorem the assumption that $(G,\cdot)$ is commutative is essentially needed. Indeed, if $(G,+)$ is an abelian group of exponent 4, then we have $p_3(G,\cdot) = 3$ for the groupoid $(G,\cdot) = (G, 2x+3y)$. Obviously $(G,\cdot)$ is a noncommutative idempotent groupoid and is not an affine space over $GF(3)$.

THEOREM 6. *Let $(G,\cdot)$ be an idempotent groupoid with $p_2(G,\cdot) = 1$ Then the following conditions are equivalent:*

(1) *$(G,\cdot)$ is an affine space over $GF(3)$;*

(2) $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ *for some* $n \geq 4$;
(3) $(G,\cdot)$ *is medial and satisfies a nonregular identity;*
(4) $p_3(G,\cdot) = 3$ *and the clone of* $(G,\cdot)$ *is minimal*

PROOF. Since $p_2(G,\cdot) = 1$, we infer that $(G,\cdot)$ is commutative and hence the equivalence (i) $\Leftrightarrow$ (ii) follows from Theorem 4. According to [5,], the groupoid $(G,\cdot)$ is either a nontrivial Steiner quasigroup or a nontrivial near-semilattice. The implication (i) $\Rightarrow$ (iii) is obvious since any affine space over $GF(3)$ is a medial Steiner quasigroup and hence $(G,\cdot)$ satisfies a nonregular identity, namely $xy^2 = x$. The implication (iii) $\Rightarrow$ (i) follows from [4] and Theorem 1 The equivalence (i) $\Leftrightarrow$ (iv) is contained in Theorem 5.

Recall that a groupoid $(G,\cdot)$ is called *totally commutative* if every essentially binary term is commutative. Further,

PROPOSITION 7 *([1]) Let $(G,\cdot)$ be an proper medial idempotent groupoid. Then the following conditions are equivalent:*

(1) $(G,\cdot)$ *is an totally commutative;*
(2) $(G,\cdot)$ *is either a semilattice or an affine space over* $GF(3)$;
(3) $p_2(G,\cdot) = 1$.

THEOREM 8 *Let $(G,\cdot)$ be a proper idempotent groupoid. Then $(G,\cdot)$ is a nonmedial distributive Steiner quasigroup if and only if $p_2(G,\cdot) = 1$, $p_3(G,\cdot) = 3$ and $p_n(G,\cdot) > 3^{n-1}$ for all $n \geq 4$.*

PROOF. If $(G,\cdot)$ is nonmedial distributive Steiner quasigroup, then by Theorem 1 we infer that $(G,\cdot)$ is not an affine space over $GF(3)$. Obviously $p_2(G,\cdot) = 1$ and $p_3(G,\ ) = 3$ for such groupoids. Using [7,Theorem 5 1], we get that $p_n(G,\cdot) > 3^{n-1}$ for all $n \geq 4$ Let now $p_2(G,\cdot) = 1$ and $p_3(G,\cdot) = 3$. Thus $(G,\ )$ is a commutative idempotent groupoid. According to [9,Theorem 1.2], the groupoid $(G,\cdot)$ is a distributive Steiner quasigroup. Since $p_n(G,\cdot) > 3^{n-1}$, we infer that $(G,\cdot)$ is nonmedial. Indeed, if $(G,\cdot)$ is medial, then the $(G,\cdot)$ is an affine space over $GF(3)$ by Theorem 1 and we will have $\frac{2^n-(-1)^n}{3} = p_n(G,\cdot) > 3^{n-1}$ for all $n \geq 4$, which is impossible. This completes the proof.

The *code* of an algebra $\mathcal{A}$ is a finite sequence $\mathbf{q} = (p_0(\mathcal{A}), \dots, p_m(\mathcal{A}))$ such that the p_n-sequence $\mathbf{p} = (p_0(\mathcal{A}), p_1(\mathcal{A}), p_2(\mathcal{A}), \dots)$ is the unique extension of $\mathbf{q}$ and m is the smallest number with this property.

THEOREM 9. *Let $(G, \cdot)$ be an nontrivial groupoid. Then the following conditions are equivalent:*

(1) *$(G, \cdot)$ is a nontrivial affine space over $GF(3)$;*
(2) *$(G, \cdot)$ represents the sequence* $\mathbf{a}= (0, 1, 1, 3, \dots, \frac{2^n-(-1)^n}{3}, \dots)$*;*
(3) *The sequence $(0, 1, 1, 3, 5)$ is the code of $(G, \cdot)$ in the class of all groupoids.*

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) is by Theorem 2. The implication (iii) $\Rightarrow$ (i) follows from the definition of the code and Theorem 2. We prove here the implication (i) $\Rightarrow$ (iii). Since semilattices also represent $(0, 1, 1)$, this sequence does not determine affine spaces over $GF(3)$. Thus $(0, 1, 1)$ is not the code of an affine space. If $(G, \cdot)$ represents $(0, 1, 1, 3)$, then $(G, \cdot)$ is a commutative idempotent groupoid and by [9,Theorem 1 2] we infer that $(G, \cdot)$ is a nontrivial distributive Steiner quasigroup. Since there exist nonmedial distributive Steiner quasigroups ([16]), obviously representing $(0, 1, 1, 3)$, we deduce by applying the preceding theorem that for such groupoids we have $p_n(G, \cdot) > 3^{n-1}$ for all $n \geq 4$. Thus these groupoids are not affine spaces over $GF(3)$ (see Theorem 2). Thus $(0, 1, 1, 3)$ is not the code of affine spaces over $GF(3)$. If $(G, \cdot)$ represents $(0, 1, 1, 3, 5)$, then $(G, \cdot)$ is a nontrivial affine space over $GF(3)$ by Theorem 4. Thus, $(0, 1, 1, 3, 5)$ is the code of affine spaces in the class of all groupoids.

Recall that an algebra A of a finite type is called *equationally complete* if the variety generated by A is equationally complete.

THEOREM 10 *Let $(G, \cdot)$ be an idempotent groupoid with $p_2(G, \cdot) = 1$. Then $(G, \cdot)$ is equationally complete if and only if $(G, \cdot)$ is either a nontrivial affine space over $GF(3)$ or a nontrivial semilattice.*

PROOF. By [5, Lemma 1] we infer that $(G, \cdot)$ is either a nontrivial Steiner quasigroup or a nontrivial near-semilattice. First observe that any nontrivial affine space over $GF(p)$ is equationally complete

([14]). Further, let $(G, \cdot)$ be a nontrivial Steiner quasigroup then it is obvious that the subgroupoid $G(a, b)$ generated by two distinct elements a, b in G is isomorphic to three-element affine space over $GF(3)$, namely it is isomorphic to the groupoid $G(3) = (\{0, 1, 2\}, 2x+2y)$ where $(\{0, 1, 2\}, +)$ is a group of order 3. Clearly, the variety V_1 generated by $G(3)$ is contained in the variety V_2 generated by $(G, \cdot)$. Since the variety generated by $GF(3)$ is equationally complete, this is precisely the variety of all affine space over $GF(3)$ and we get that $V_1 = V_2$ provided V_2 is equationally complete. Analogously, any nontrivial near-semilattice $(G, \cdot)$ contains a two-element semilattice and therefore if $(G, \cdot)$ is equationally complete then $(G, \cdot)$ must be a semilattice, completing the proof of the theorem.

In [3], we find the following.

PROPOSITION 11. *Let (G, f) be a nontrivial symmetric algebra of type (4) satisfying the identity $f(x, y, y, y) = x$. Then (G, f) is a nontrivial affine space over $GF(3)$ if and only if $p_4(G, f) = 5$.*

Combining some earlier results we have the following.

THEOREM 12. *Let $(G, \cdot)$ be an idempotent groupoid with $p_2(G, \cdot) = 1$. Then the following conditions are equivalent*

(1) *$(G, \cdot)$ is a nontrivial affine space over $GF(3)$;*
(2) *$p_4(G, \cdot) = 5$ (without the assumption $p_2(G, \cdot) = 1$;*
(3) *$p_3(G, \cdot) = 3$ and the clone of $(G, \cdot)$ is minimal;*
(4) *$(G, \cdot)$ is equationally complete and $p_n(G, \cdot) > 1$ for some $n \geq 3$;*
(5) *$(G, \cdot)$ is equationally complete and $(G, \cdot)$ satisfies a nonregular identity;*
(6) *$(G, \cdot)$ satisfies a nontrivial strongly regular identity and a nonregular identity.*

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) is contained in Proposition 3. The equivalence (i) $\Leftrightarrow$ (iii) follows from Theorem 5. The equivalence (i) $\Leftrightarrow$ (iv) can be deduced from Theorem 10. Using the same argument as in the proof of Theorem 10, one can obtain the equivalence (iv) $\Leftrightarrow$ (v). We prove the equivalence (i) $\Leftrightarrow$ (vi). It is clear that any nontrivial affine space over $GF(3)$ satisfies a strongly regular identity, e.g., the medial law, and it also satisfies a nonregular identity, e.g., $xy^2 = x$. The converse follows from [4], [7] and Theorem 1.

THEOREM 13. *If an idempotent algebra (A,Ω) with $p_2(A,\Omega) \geq 2$ contains a Steiner quasigroup as a reduct, then $p_2(A,\Omega) \geq 5$*

PROOF. Suppose $(A,+)$ is such a reduct of (A,Ω). Since $p_2(A,\Omega) \geq 2$, we infer that (A,Ω) contains another essentially binary term, say $x \cdot y$. If $x \cdot y$ is commutative, then we prove that the terms $x+y$, xy, $(x+y)+xy$, $xy+y$ and $yx+x$ are pairwise distinct essentially binary terms. Indeed, if for example $xy+y=y$, then we have $y = y+y = (xy+y)+y = xy$, a contradiction. If $xy+y=x$ then we also have the contradiction that $xy = x+y$. If $xy+y = yx+x$ then we obtain the contradiction $x=y$, and so on. Thus we have that $p_2(A,\Omega) \geq (A,+,\cdot) \geq 5$. Assume that $x \cdot y$ is noncommutative. Then we consider the terms $x+y$, xy, yx, $xy+y$ and $yx+x$. By the same argument as above, we see that $xy+y$ is essentially binary and $xy+y \neq xy$. If $xy+y=yx$, then $xy+yx=y$, which is a contradiction. Obviously $x+y \neq xy+y$. Assume $xy+x = yx+x$. Then we consider the following essentially binary terms $x+y$, xy, yx, $xy+y$ and $(x+y)+(xy+y)$, and we see that they are pairwise distinct. Thus $p_2(A,\Omega) \geq (A,+,\cdot) \geq 5$ in this case as well, which completes the proof.

3. Appendix

We summarize here all known characterizations of affine space over $GF(3)$ in a list.

For a groupoid $(G,\cdot)$ the following conditions are equivalent:

(1) $(G,\cdot)$ is an affine space over $GF(3)$,
(2) $(G,\cdot)$ represent the sequence $(0,1,1,3,\ldots,\frac{2^n-(-1)^n}{3},\ldots)$,
(3) The sequence $(0,1,1,3,5)$ is the code of $(G,\cdot)$ in the class of all groupoids,
(4) $(G,\cdot)$ is idempotent and $p_4(G,\cdot)=5$,
(5) $(G,\cdot)$ is commutative, idempotent and $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ for some $n \geq 4$,

(6) $(G,\cdot)$ is a nontrivial medial Steiner quasigroup;
(7) $(G,\cdot)$ is a Steiner quasigroup whose clone is minimal;
(8) $(G,\cdot)$ is a nontrivial Steiner quasigroup in which $((x_1x_2)(x_3x_4))x5$ is symmetric;
(9) $(G,\cdot)$ is a nontrivial Steiner quasigroup satisfying a nontrivial linear identity;
(10) $(G,\cdot)$ is commutative, $p_3(G,\cdot) = 3$ and the clone of $(G,\cdot)$ is minimal;
(11) $p_2(G,\cdot) = 1$ and $(G,\cdot)$ is medial satisfying a nonregular identity;
(12) $p_2(G,\cdot) = 1$ and $(G,\cdot)$ is satisfies both a nonregular identity and a nontrivial strongly regular identity,
(13) $p_2(G,\cdot) = 1$, $p_n(G,\cdot) > 1$ for some $n > 1$ and the clone of $(G,\cdot)$ is minimal,
(14) $p_2(G,\cdot) = 1$, $(G,\cdot)$ satisfies a nonregular identity and the clone of $(G,\cdot)$ is minimal;
(15) $p_2(G,\ ) = 1$, $(G,\cdot)$ is equationally complete and $p_n(G,\cdot) > 1$ for some $n \geq 3$,
(16) $p_2(G,\cdot) = 1$, $(G,\cdot)$ is equationally complete and $(G,\cdot)$ satisfies a nonregular identity;
(17) $p_2(G,\cdot) = 1$, $(G,\cdot)$ is medial, idempotent, but not a semilattice;
(18) $p_3(G,\cdot) = 3$, $(G,\cdot)$ is commutative, idempotent and equationally complete;
(19) $(G,\cdot)$ is medial idempotent totally commutative groupoid which is not a semilattice;
(20) $p_3(G,\ ) < 7$, $(G,\cdot)$ is not a semilattice and the clone of $(G,\cdot)$ is minimal,
(21) $(G,\cdot)$ is a commutative idempotent groupoid which is not a semilattice and every term over $(G,\cdot)$ is equal to a linear term,
(22) $(G,\cdot)$ is idempotent and equationally complete with $p_3(G,\cdot) \leq 6$;
(23) $(G,\cdot)$ is a nontrivial Steiner quasigroup and $p_4(G,\cdot) \leq 35$,
(24) $(G,\ )$ is a nontrivial Steiner quasigroup with $p_n(G,\cdot) < \frac{7}{8}n!$ for some $n \geq 5$;
(25) (G,f) is a symmetric algebra of type (4) satisfying $f(x,y,y,y) = x$ and $p_4(G,f) \geq 5$.

In this connection, we raise the following problems.

PROBLEM 1. *Let $(G,\cdot)$ be an idempotent groupoid. Is it true that $(G,\cdot)$ is an affine space over $GF(3)$ if and only if $p_n(G,\cdot) = \frac{2^n-(-1)^n}{3}$ for some $n \geq 4$.*

PROBLEM 2. *Let $(G,\cdot)$ be an idempotent groupoid which is equationally complete. Examine p_n-sequences of such groupoids. Note that there exists no equationally complete idempotent groupoid with $p_3(G,\cdot) = 6$.*

REFERENCES

[1] J. R. Cho and S. H. Kim, *Totally commutative idempotent groupoids, preprint.*.
[2] B. Csákány, *On affine spaces over prime fields*, Acta Sci. Math. **37** (1975), 33-36.
[3] J. Dudek, *An affine space over $GF(3)$ as a 4-groupoid*, Proc. Symposium on n-ary structures, Skopje, 1982, pp. 243-249
[4] J. Dudek, *Varieties of idempotent commutative groupoids*, Fund. Math. **120** (1984), 193-204
[5] J. Dudek, *On the minimal extension of sequences*, Algebra Universalis **23** (1986), 308-312.
[6] J. Dudek, *A polynomial characterization of affine spaces over $GF(3)$*, Colloq. Math. **50** (1986), 161-171
[7] J. Dudek, *Polynomials in idempotent commutative groupoids*, Dissertationes Math. **286** (1989), 1-55
[8] J. Dudek, *The unique minimal clone with three essentially binary operations*, Algebra Universalis **27** (1990), 201-209
[9] J. Dudek, *A characterization of distributive Steiner quasigroups and semilattices*, Discuss. Math., Algebra and Stochastic Methods **15** (1995), 101-119
[10] G. Gratzer, *Universal Algebra, 2nd ed.*, Springer-Verlag, New York-Heidelberg-Berlin, 1979
[11] G. Grätzer and A. Kisielewicz,, *A survey of some open problems on p_n-sequences and free spectra of algebras and varieties*, Universal Algebra and Quasigroup Theory, A. Romanowska and J. D. H. Smith (eds) (1992), Helderman Verlag, Berlin, 57-88
[12] G. Gratzer and R. Padmanabhan,, *On commutative, idempotent and nonassociative groupoids*, Proc. Amer. Math. Soc. **28** (1971), 75-78
[13] F. Ostermann and J. Schmidt,, *Der baryzentrische Kalkul als axiomtische Grundlage der affinen Geometire*, J. reine angew. Math. **224** (1966), 44-57
[14] J. Płonka, *R-prime idempotent reduct of abelian groups*, Archiv der Math. **24** (1973), 129-132
[15] A. Romanowska and J. D. H. Smith, *Modal Theory, an algebraic approach to order, geometry and convexity*, Helderman Verlag, Berlin, 1985.

[16] J Soublin, *Etude algebrique de la notion de moyenne*, J Math. Pures et App **50** (1971), 53-264
[17] Á Szendrei, *Clones in universal algebra*, (Université de Montréal, Montréal, 1986)

Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: jungcho@hyowon.pusan.ac.kr