ON $p_n$-SEQUENCES OF UNIVERSAL ALGEBRAS

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ABSTRACT We study how the $p_n$-sequence of a universal algebra determine the structure of the algebra. Regarding term equivalent algebras as the same algebras, we consider the problem when the algebras are groupoids.

1. Introduction

A term $f(x_1, x_2, \ldots, x_n)$ over an abstract algebra $A = (A, \Omega)$ is called $n$-ary if it involves $n$ distinct variables and essential if it depends on each variable it involves in the sense that, for each $i = 1, 2, \ldots, n$, there are $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ and $b, c$ in $A$ such that

$$f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n).$$

We denote by $p_n(A)$ the number of essentially $n$-ary term functions over $A$, and the sequence $(p_0(A), p_1(A), p_2(A), \ldots)$ is called the $p_n$-sequence of $A$.

A groupoid is called trivial if it has only one element and proper if the term $xy$ is essentially binary.

Two algebras $(A, \Omega_1)$ and $(A, \Omega_2)$ on the same underlying set $A$ are said to be term equivalent if they have the same term functions, that is, any $\Omega_1$-term can be written as an $\Omega_2$-term and vice versa.

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For simplicity of our notation, we inductively define groupoid terms by \( xy^1 = xy \) and \( xy^{k+1} = (xy^k)y \), and use the expression \( x_1x_2 \cdots x_{n-1}x_n \) for \((\cdots(x_1x_2)\cdots)x_{n-1})x_n\).

A groupoid \((G, \cdot)\) is said to be medial if it satisfies the identity \((xy)(uv) = (xu)(yv)\), and distributive if it satisfies the \(x(yz) = (xy)(xz)\) and \((xy)z = (xz)(yz)\). A commutative idempotent groupoid is called a semilattice if it is a semigroup, a near-semilattice if it satisfies \(xy^2 = xy\), and a Steiner quasigroup if it satisfies \(xy^2 = x\).

An affine space over a field \(K\) is algebraically defined to be the full idempotent reduct of a vector space over \(K\) ([2,13,15,17]). However, when the base field is the Galois field \(GF(3)\) with three element, any affine space over \(GF(3)\) is term equivalent to a medial Steiner quasigroup ([12]). Thus we will treat an affine space over \(GF(3)\) as a medial Steiner quasigroup in this paper.

We say a sequence \(a = (a_0, a_1, a_2, \ldots)\) (finite or infinite) of cardinals is called representable if there is an algebra \(A\) such that \(p(A) = a\), that is, \(p_n(A) = a_n\) for all \(n\), and call \(a\) the \(p_n\)-sequence of \(A\) in this case. If, furthermore, \(A\) is from a given class \(K\) of algebra, we say that \(a\) is representable in \(K\) or \(A\) represents \(a\) in \(K\).

A clone on a set \(A\) is a collection of operations on \(A\) which is closed under compositions and contains all projections. A clone \(C\) is called minimal if the lattice of subclones of \(C\) has only two elements. This means that \(\text{Card}(C) \neq 1\) and any term in \(C\) together with projections generates \(C\).

For further concepts and notations not defined in this paper, we refer the readers to [10] and [11].

A term \(f(x_1, x_2, \ldots, x_n)\) over a groupoid \((G, \cdot)\) will be called linear term if each variable appears at most once in the expression.

2. Theorems and proofs

**Theorem 1.** Let \((G, \cdot)\) be a nontrivial Steiner quasigroup. Then the following conditions are equivalent:

1. \((G, \cdot)\) is an affine space over \(GF(3)\);
2. \((G, \cdot)\) is medial;
The clone of \((G, \cdot)\) is minimal;

4. The term \(f(x_1, x_2, x_3, x_4, x_5) = ((x_1 x_2)(x_3 x_4)) x_5\) is symmetric;

5. For a certain \(n \geq 4\), an \(n\)-ary term admits a nontrivial permutation.

**Proof.** The fact that the condition (i) implies any of the remaining one is not hard to check except the implication (i) \(\Rightarrow\) (iii), which can be deduced from [14]. The implication (ii) \(\Rightarrow\) (i) is contained in [12]. Using [8, Lemma 3.2], one can easily prove (iii) \(\Rightarrow\) (ii). Now we prove (iv) \(\Rightarrow\) (i) If \(f(x_1, \cdots, x_5)\) is symmetric, then we obtain that

\[
\begin{align*}
(x_1 x_2)(x_3 x_4) &= ((x_1 x_2)(x_3 x_4))((x_1 x_2)(x_3 x_4)) \\
&= ((x_1 x_3)(x_2 x_4))((x_1 x_2)(x_3 x_4)) \\
&= ((x_1 x_2)(x_3 x_4))((x_1 x_3)(x_2 x_4)) \\
&= ((x_1 x_3)(x_2 x_4))((x_1 x_3)(x_2 x_4)) \\
&= (x_1 x_3)(x_2 x_4).
\end{align*}
\]

The implication (v) \(\Rightarrow\) (ii) follows from [4, Theorem 4].

Now we recall a theorem of Gátzer and Padmanabhan from [12].

**Proposition 2.** If \(A\) is an algebra, then \(A\) is a nontrivial affine space over \(GF(3)\) if and only if \(p_n(A) = \frac{2^n - (-1)^n}{3}\) for all \(n\). Moreover, if \(A\) is a groupoid, then it suffices to assume \(p_n(A) = \frac{2^n - (-1)^n}{3}\) only for \(n = 0, 1, 2, 3, 4\).

With this theorem is connected

**Proposition 3.** ([6, Theorem]) Let \((G, \cdot)\) be an idempotent groupoid. Then \((G, \cdot)\) is a nontrivial affine space over \(GF(3)\) if and only if \(p_4(G, \cdot) = 5\).

Note that there exist idempotent groupoids \((G, \cdot)\) satisfying \(p_n(A) = \frac{2^n - (-1)^n}{3}\) for all \(n \leq 3\) which are not affine spaces over \(GF(3)\) ([16]). This means that \(p_4(G, \cdot) = 5\) is the first number of the \(p_n\)-sequence which uniquely determines the structure of an idempotent groupoid, and such groupoids are affine spaces over \(GF(3)\) (see Theorem 9).
Theorem 4. Let \((G, \cdot)\) be a commutative idempotent groupoid. Then \((G, \cdot)\) is a nontrivial affine space over \(GF(3)\) if and only if \(p_n(G, \cdot) = \frac{2^n - (1)^n}{3}\) for some \(n \geq 4\).

Proof. If \((G, \cdot)\) is a nontrivial affine space over \(GF(3)\), i.e., \((G, \cdot) = (G, 2x + 2y)\), where \((G, +)\) is an abelian group of exponent 3, then \(p_n(G, \cdot) = \frac{2^n - (1)^n}{3}\) for all \(n\) by [2]. Let now \(p_n(G, \cdot) = \frac{2^n - (1)^n}{3}\) for some \(n \geq 4\). Then \((G, \cdot)\) is not a semilattice since \(p_n(G, \cdot) = 1\) for all \(n\) if \((G, \cdot)\) is a semilattice. If \((G, \cdot)\) is also not an affine space over \(GF(3)\), then by [7, Theorem 1] we obtain that \(p_n(G, \cdot) \geq 3^{n-1}\) for all \(n \geq 4\). Hence, \(\frac{2^n - (1)^n}{3} \geq 3^n\) for all \(n \geq 4\), which is not true. Thus \((G, \cdot)\) is an affine space over \(GF(3)\).

In this connection we conjecture that if \((G, \cdot)\) is an idempotent groupoid (not necessarily commutative), then \((G, \cdot)\) is a nontrivial affine space over \(GF(3)\) if and only if \(p_n(G, \cdot) = \frac{2^n - (1)^n}{3}\) for some \(n \geq 4\) (compare with Theorem 6).

Theorem 5. Let \((G, \cdot)\) be a commutative idempotent groupoid. Then \((G, \cdot)\) is a nontrivial affine space over \(GF(3)\) if and only if \(p_3(G, \cdot) = 3\) and the clone of \((G, \cdot)\) is minimal.

Proof. If \((G, \cdot)\) is a nontrivial affine space over \(GF(3)\), then trivially \(p_3(G, \cdot) = 3\) and the clone of \((G, \cdot)\) is minimal by [14]. Assume that \(p_3(G, \cdot) = 3\) and the clone of \((G, \cdot)\) is minimal. By [9, Theorem 1.2], \((G, \cdot)\) is a nontrivial distributive Steiner quasigroup. Then by Theorem 1 \((G, \cdot)\) is an affine space over \(GF(3)\).

Note that in this theorem the assumption that \((G, \cdot)\) is commutative is essentially needed. Indeed, if \((G, +)\) is an abelian group of exponent 4, then we have \(p_3(G, \cdot) = 3\) for the groupoid \((G, \cdot) = (G, 2x + 3y)\). Obviously \((G, \cdot)\) is a noncommutative idempotent groupoid and is not an affine space over \(GF(3)\).

Theorem 6. Let \((G, \cdot)\) be an idempotent groupoid with \(p_2(G, \cdot) = 1\). Then the following conditions are equivalent:

1. \((G, \cdot)\) is an affine space over \(GF(3)\);
(2) $p_n(G, \cdot) = \frac{2^n - (-1)^n}{3}$ for some $n \geq 4$;
(3) $(G, \cdot)$ is medial and satisfies a nonregular identity;
(4) $p_3(G, \cdot) = 3$ and the clone of $(G, \cdot)$ is minimal

**Proof.** Since $p_2(G, \cdot) = 1$, we infer that $(G, \cdot)$ is commutative and hence the equivalence (i) $\Leftrightarrow$ (ii) follows from Theorem 4. According to [5], the groupoid $(G, \cdot)$ is either a nontrivial Steiner quasigroup or a nontrivial near-semilattice. The implication (i) $\Rightarrow$ (iii) is obvious since any affine space over $GF(3)$ is a medial Steiner quasigroup and hence $(G, \cdot)$ satisfies a nonregular identity, namely $xy^2 = x$. The implication (iii) $\Rightarrow$ (i) follows from [4] and Theorem 1. The equivalence (i) $\Leftrightarrow$ (iv) is contained in Theorem 5.

Recall that a groupoid $(G, \cdot)$ is called *totally commutative* if every essentially binary term is commutative. Further,

**Proposition 7** ([11]) Let $(G, \cdot)$ be an proper medial idempotent groupoid. Then the following conditions are equivalent:

1. $(G, \cdot)$ is an totally commutative;
2. $(G, \cdot)$ is either a semilattice or an affine space over $GF(3)$;
3. $p_2(G, \cdot) = 1$.

**Theorem 8** Let $(G, \cdot)$ be a proper idempotent groupoid. Then $(G, \cdot)$ is a nonmedial distributive Steiner quasigroup if and only if $p_2(G, \cdot) = 1$, $p_3(G, \cdot) = 3$ and $p_n(G, \cdot) > 3^{n-1}$ for all $n \geq 4$.

**Proof.** If $(G, \cdot)$ is nonmedial distributive Steiner quasigroup, then by Theorem 1 we infer that $(G, \cdot)$ is not an affine space over $GF(3)$. Obviously $p_2(G, \cdot) = 1$ and $p_3(G, \cdot) = 3$ for such groupoids. Using [7, Theorem 5.1], we get that $p_n(G, \cdot) > 3^{n-1}$ for all $n \geq 4$. Let now $p_2(G, \cdot) = 1$ and $p_3(G, \cdot) = 3$. Thus $(G, \cdot)$ is a commutative idempotent groupoid. According to [9, Theorem 1.2], the groupoid $(G, \cdot)$ is a distributive Steiner quasigroup. Since $p_n(G, \cdot) > 3^{n-1}$, we infer that $(G, \cdot)$ is nonmedial. Indeed, if $(G, \cdot)$ is medial, then the $(G, \cdot)$ is an affine space over $GF(3)$ by Theorem 1 and we will have $2^n - (-1)^n = p_n(G, \cdot) > 3^{n-1}$ for all $n \geq 4$, which is impossible. This completes the proof.
The code of an algebra $\mathcal{A}$ is a finite sequence $q = (p_0(\mathcal{A}), \ldots, p_m(\mathcal{A}))$ such that the $p_n$-sequence $p = (p_0(\mathcal{A}), p_1(\mathcal{A}), p_2(\mathcal{A}), \ldots)$ is the unique extension of $q$ and $m$ is the smallest number with this property.

**Theorem 9.** Let $(G, \cdot)$ be a nontrivial groupoid. Then the following conditions are equivalent:

1. $(G, \cdot)$ is a nontrivial affine space over $GF(3)$;
2. $(G, \cdot)$ represents the sequence $a = (0, 1, 1, 3, \ldots, \frac{2^n-(-1)^n}{3}, \ldots)$;
3. The sequence $(0, 1, 1, 3, 5)$ is the code of $(G, \cdot)$ in the class of all groupoids.

**Proof.** The equivalence $(i) \iff (ii)$ is by Theorem 2. The implication $(iii) \Rightarrow (i)$ follows from the definition of the code and Theorem 2. We prove here the implication $(i) \Rightarrow (iii)$. Since semilattices also represent $(0, 1, 1)$, this sequence does not determine affine spaces over $GF(3)$. Thus $(0, 1, 1)$ is not the code of an affine space. If $(G, \cdot)$ represents $(0, 1, 1, 3)$, then $(G, \cdot)$ is a commutative idempotent groupoid and by [9, Theorem 1.2] we infer that $(G, \cdot)$ is a nontrivial distributive Steiner quasigroup. Since there exist nonmedial distributive Steiner quasigroups ([16]), obviously representing $(0, 1, 1, 3)$, we deduce by applying the preceding theorem that for such groupoids we have $p_n(G, \cdot) > 3^{n-1}$ for all $n \geq 4$. Thus these groupoids are not affine spaces over $GF(3)$ (see Theorem 2). Thus $(0, 1, 1, 3)$ is not the code of affine spaces over $GF(3)$. If $(G, \cdot)$ represents $(0, 1, 1, 3, 5)$, then $(G, \cdot)$ is a nontrivial affine space over $GF(3)$ by Theorem 4. Thus, $(0, 1, 1, 3, 5)$ is the code of affine spaces in the class of all groupoids.

Recall that an algebra $\mathcal{A}$ of a finite type is called *equationally complete* if the variety generated by $\mathcal{A}$ is equationally complete.

**Theorem 10** Let $(G, \cdot)$ be an idempotent groupoid with $p_2(G, \cdot) = 1$. Then $(G, \cdot)$ is equationally complete if and only if $(G, \cdot)$ is either a nontrivial affine space over $GF(3)$ or a nontrivial semilattice.

**Proof.** By [5, Lemma 1] we infer that $(G, \cdot)$ is either a nontrivial Steiner quasigroup or a nontrivial near-semilattice. First observe that any nontrivial affine space over $GF(p)$ is equationally complete.
Further, let \( (G, \cdot) \) be a nontrivial Steiner quasigroup then it is obvious that the subgroupoid \( G(a, b) \) generated by two distinct elements \( a, b \) in \( G \) is isomorphic to three-element affine space over \( GF(3) \), namely it is isomorphic to the groupoid \( G(3) = \{0, 1, 2\}, 2x + 2y \) where \( (\{0, 1, 2\}, +) \) is a group of order 3. Clearly, the variety \( V_1 \) generated by \( G(3) \) is contained in the variety \( V_2 \) generated by \( (G, \cdot) \). Since the variety generated by \( GF(3) \) is equationally complete, this is precisely the variety of all affine space over \( GF(3) \) and we get that \( V_1 = V_2 \) provided \( V_2 \) is equationally complete. Analogously, any nontrivial near-semilattice \( (G, \cdot) \) contains a two-element semilattice and therefore if \( (G, \cdot) \) is equationally complete then \( (G, \cdot) \) must be a semilattice, completing the proof of the theorem.

In [3], we find the following.

**Proposition 11.** Let \( (G, f) \) be a nontrivial symmetric algebra of type \( (4) \) satisfying the identity \( f(x, y, y, y) = x \). Then \( (G, f) \) is a nontrivial affine space over \( GF(3) \) if and only if \( p_4(G, f) = 5 \).

Combining some earlier results we have the following.

**Theorem 12.** Let \( (G, \cdot) \) be an idempotent groupoid with \( p_2(G, \cdot) = 1 \). Then the following conditions are equivalent

1. \( (G, \cdot) \) is a nontrivial affine space over \( GF(3) \);
2. \( p_4(G, \cdot) = 5 \) (without the assumption \( p_2(G, \cdot) = 1 \);
3. \( p_3(G, \cdot) = 3 \) and the clone of \( (G, \cdot) \) is minimal;
4. \( (G, \cdot) \) is equationally complete and \( p_n(G, \cdot) > 1 \) for some \( n \geq 3 \);
5. \( (G, \cdot) \) is equationally complete and \( (G, \cdot) \) satisfies a nonregular identity;
6. \( (G, \cdot) \) satisfies a nontrivial strongly regular identity and a nonregular identity.

**Proof.** The equivalence \( (i) \Leftrightarrow (ii) \) is contained in Proposition 3. The equivalence \( (i) \Leftrightarrow (iii) \) follows from Theorem 5. The equivalence \( (i) \Leftrightarrow (iv) \) can be deduced from Theorem 10. Using the same argument as in the proof of Theorem 10, one can obtain the equivalence \( (iv) \Leftrightarrow (v) \). We prove the equivalence \( (i) \Leftrightarrow (vi) \). It is clear that any nontrivial affine space over \( GF(3) \) satisfies a strongly regular identity, e.g., the
medial law, and it also satisfies a nonregular identity, e.g., $xy^2 = x$. The converse follows from [4], [7] and Theorem 1.

**Theorem 13.** If an idempotent algebra $(A, \Omega)$ with $p_2(A, \Omega) \geq 2$ contains a Steiner quasigroup as a reduct, then $p_2(A, \Omega) \geq 5$.

**Proof.** Suppose $(A, +)$ is such a reduct of $(A, \Omega)$. Since $p_2(A, \Omega) \geq 2$, we infer that $(A, \Omega)$ contains another essentially binary term, say $x \cdot y$. If $x \cdot y$ is commutative, then we prove that the terms $x + y$, $xy$, $(x + y) + xy$, $xy + y$ and $yx + x$ are pairwise distinct essentially binary terms. Indeed, if for example $xy + y = y$, then we have $y = y + y = (xy + y) + y = xy$, a contradiction. If $xy + y = x$ then we also have the contradiction that $xy = x + y$. If $xy + y = yx + x$ then we obtain the contradiction $x = y$, and so on. Thus we have that $p_2(A, \Omega) \geq (A, +, \cdot) \geq 5$. Assume that $x \cdot y$ is noncommutative. Then we consider the terms $x + y$, $xy$, $yx$, $xy + y$ and $yx + x$. By the same argument as above, we see that $xy + y$ is essentially binary and $xy + y \neq xy$. If $xy + y = yx$, then $xy + yx = y$, which is a contradiction. Obviously $x + y \neq xy + y$. Assume $xy + x = yx + x$. Then we consider the following essentially binary terms $x + y$, $xy$, $yx$, $xy + y$ and $(x + y) + (xy + y)$, and we see that they are pairwise distinct. Thus $p_2(A, \Omega) \geq (A, +, \cdot) \geq 5$ in this case as well, which completes the proof.

3. Appendix

We summarize here all known characterizations of affine space over $GF(3)$ in a list.

For a groupoid $(G, \cdot)$ the following conditions are equivalent:

1. $(G, \cdot)$ is an affine space over $GF(3)$,
2. $(G, \cdot)$ represent the sequence $(0, 1, 1, 3, \ldots, \frac{2^n - (-1)^n}{3}, \ldots)$,
3. The sequence $(0, 1, 1, 3, 5)$ is the code of $(G, \cdot)$ in the class of all groupoids,
4. $(G, \cdot)$ is idempotent and $p_4(G, \cdot) = 5$,
5. $(G, \cdot)$ is commutative, idempotent and $p_n(G, \cdot) = \frac{2^n - (-1)^n}{3}$ for some $n \geq 4$. 
(6) \((G, \cdot)\) is a nontrivial medial Steiner quasigroup;
(7) \((G, \cdot)\) is a Steiner quasigroup whose clone is minimal;
(8) \((G, \cdot)\) is a nontrivial Steiner quasigroup in which \(((x_1x_2)(x_3x_4)x_5\) is symmetric;
(9) \((G, \cdot)\) is a nontrivial Steiner quasigroup satisfying a nontrivial linear identity;
(10) \((G, \cdot)\) is commutative, \(p_3(G, \cdot) = 3\) and the clone of \((G, \cdot)\) is minimal;
(11) \(p_2(G, \cdot) = 1\) and \((G, \cdot)\) is medial satisfying a nonregular identity;
(12) \(p_2(G, \cdot) = 1\) and \((G, \cdot)\) is satisfies both a nonregular identity and a nontrivial strongly regular identity,
(13) \(p_2(G, \cdot) = 1, p_n(G, \cdot > 1\) for some \(n > 1\) and the clone of \((G, \cdot)\) is minimal,
(14) \(p_2(G, \cdot) = 1, (G, \cdot)\) satisfies a nonregular identity and the clone of \((G, \cdot)\) is minimal;
(15) \(p_2(G, \cdot) = 1, (G, \cdot)\) is equationally complete and \(p_n(G, \cdot) > 1\) for some \(n \geq 3\),
(16) \(p_2(G, \cdot) = 1, (G, \cdot)\) is equationally complete and \((G, \cdot)\) satisfies a nonregular identity;
(17) \(p_2(G, \cdot) = 1, (G, \cdot)\) is medial, idempotent, but not a semilattice;
(18) \(p_3(G, \cdot) = 3, (G, \cdot)\) is commutative, idempotent and equationally complete;
(19) \((G, \cdot)\) is medial idempotent totally commutative groupoid which is not a semilattice;
(20) \(p_3(G, \cdot) < 7, (G, \cdot)\) is not a semilattice and the clone of \((G, \cdot)\) is minimal,
(21) \((G, \cdot)\) is a commutative idempotent groupoid which is not a semilattice and every term over \((G, \cdot)\) is equal to a linear term,
(22) \((G, \cdot)\) is idempotent and equationally complete with \(p_3(G, \cdot) \leq 6\);
(23) \((G, \cdot)\) is a nontrivial Steiner quasigroup and \(p_4(G, \cdot) \leq 35\),
(24) \((G, \cdot)\) is a nontrivial Steiner quasigroup with \(p_n(G, \cdot) < \frac{7}{8} n!\) for some \(n \geq 5\);
(25) \((G, f)\) is a symmetric algebra of type \((4)\) satisfying \(f(x, y, y, y) = x\) and \(p_4(G, f) \geq 5\).

In this connection, we raise the following problems.

**Problem 1.** Let $(G, \cdot)$ be an idempotent groupoid. Is it true that $(G, \cdot)$ is an affine space over $GF(3)$ if and only if $p_n(G, \cdot) = 2^n(-1)^n$ for some $n \geq 4$.

**Problem 2.** Let $(G, \cdot)$ be an idempotent groupoid which is equationally complete. Examine $p_n$-sequences of such groupoids. Note that there exists no equationally complete idempotent groupoid with $p_3(G, \cdot) = 6$.

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