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PERMUTING TRI-DERIVATIONS IN PRIME AND SEMI-PRIME RINGS

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ABSTRACT. In this work, we study permuting tri-derivations and give an example

1. Introduction

Throughout this work, R will represent an associative ring and Z will denote the center of R. We shall write $[x, y]$ for $xy - yx$.

A mapping $D(.,.) : R \times R \rightarrow R$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in R$. A mapping $d : R \rightarrow$ R defined by $d(x) = D(x, x)$ is called trace of $D(.,.)$, where $D(.,.) : R \times R \rightarrow R$ is a symmetric mapping It is obvious that, if $D(.,.) : R \times R \rightarrow R$ is a symmetric mapping which is also bi-additive (i e , additive in both arguments), then trace of $D(.,.)$ satisfies the relation $d(x+y) = d(x) + d(y) + 2D(x, y)$ for all $x, y \in R$.

A symmetric bi-additive mapping $D(.,.) : R \times R \rightarrow R$ is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + xD(y, z)$ is fulfilled for all $x, y, z \in R$. Then the relation $D(x, yz) = D(x, y)z + yD(x, z)$ is also fulfilled for all $x, y, z \in R$.

We shall need the following well-known and frequently used lemmas.

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LEMMA 1. *([2, Lemma 2. (ii)]) Let R be a prime ring, $a \in R$ and $d : R \to R$ an α-derivation. If U is a non-zero ideal of R and $ad(U) = 0$, then $a = 0$ or $d = 0$.*

LEMMA 2. *([11, Lemma 1]) Let R be a 2-torsion free semi-prime ring, U a non-zero ideal of R and a, b be fixed elements of R. Then the following conditions are equivalent :*

i) *$axb = 0$ for all $x \in U$,*
ii) *$bxa = 0$ for all $x \in U$,*
iii) *$axb + bxa = 0$ for all $x \in U$.*

If one of these conditions is fulfilled and $l(U) = 0$, then either $a = 0$ or $b = 0$ too, where $l(U)$ is the left annihilator of U.

2. The Results

We shall start our investigation of permuting tri-derivations with the following result.

DEFINITION 3. Let R be a ring. A mapping $D(.,.,.) : R \times R \times R \to R$ is called tri-additive if

$$D(x + w, y, z) = D(x, y, z) + D(w, y, z)$$
$$D(z, y + w, z) = D(x, y, z) + D(x, w, z)$$
$$D(x, y, z + w) = D(x, y, z) + D(x, y, w)$$

holds for all $x, y, z, w \in R$. A tri-additive mapping $D(.,.,.) : R \times R \times R \to R$ is called permuting tri-additive if $D(x, y, z) = D(x, z, y) = D(z, x, y) = D(z, y, x) = D(y, z, x) = D(y, x, z)$ holds for all $x, y, z, w \in R$. A mapping $d : R \to R$ defined by $d(x) = D(x, x, x)$ is called trace of $D(.,.,.)$, where $D(.,.,.) \cdot R \times R \times R \to R$ is a permuting tri-additive mapping

It is obvious that, if $D(.,.,.) : R \times R \times R \to R$ is a permuting tri-additive mapping then the trace of $D(.,.,.)$ satisfies the relation $d(x + y) = d(x) + d(y) + 3D(x, x, y) + 3D(x, y, y)$ for all $x, y \in R$.

A permuting tri-additive mapping $D(.,.,.) : R \times R \times R \to R$ is called a permuting tri-derivation if $D(xw, y, z) = D(x, y, z)w + xD(w, y, z)$ is fulfilled for all $x, y, z, w \in R$. Then relation $D(x, yw, z) = D(x, y, z)w + yD(x, w, z)$ and $D(x, y, zw) = D(x, y, z)w + zD(x, y, w)$ are fulfilled for all $x, y, z, w \in R$.

Let $D(.,.,.)$ be a permuting tri-additive mapping of R, where R is a ring. Since $D(0, x, y) = D(0 + 0, x, y) = D(0, x, y) + D(0, x, y)$, in this case, $D(0, x, y) = 0$ is fulfilled for all $x, y \in R$. Thus, $0 = D(0, y, z) = D(-x + x, y, z) = D(-x, y, z) + D(x, y, z)$ for all $x, y, z \in R$ and so we get that $D(-x, y, z) = -D(x, y, z)$ for all $x, y, z \in R$. Therefore, the mapping $d\ R \to R$ defined by $d(x) = D(x, x, x)$ is an odd function.

EXAMPLE 4. For a commutative ring, let

$$M = \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c, \in \mathbb{R} \right\},$$

it is obvious that M is a ring under matrix addition and multiplication. $D(.,.,.) : M \times M \times M \to M$, defined by

$$\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \to$$

$$D\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & a_1a_2a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a permuting tri-derivation.

LEMMA 5 *Let R be a 2, 3-torsion free ring, $D(.,.,.)$ a permuting tri-additive mapping of R and d the trace of $D(.,.,.)$. If $d(x) = 0$ for all $x \in R$, then $D = 0$.*

PROOF. For any $x, y \in R$,

$$d(x + y) = d(x) + d(y) + 3D(x, x, y) + 3D(x, y, y)$$

and so, from the hypothesis and since R is 3-torsion free we get, for all $x, y \in R$

$$D(x, x, y) + D(x, y, y) = 0.$$

In this case, writing $-x$ for x in this relation we have, for all $x, y \in R$

$$D(x,x,y) + D(x,y,y) = 0.$$

By adding the relation above and this last relation and since R is 2-torsion free we have $D(x,x,y) = 0$ for all $x, y, \in R$. Thus, writing $x+z, z \in R$ for x in this relation and since R is 2-torsion free we have $D(x,y,z) = 0$ for all $x, y, z \in R$. Thus, we get that $D = 0$.

Remark 6. Let R be a ring and $D(.,.,.)$ be a permuting tri-derivation of R. In this case, for any fixed $a \in R$ and for all $x, y \in R$, a mapping $D_1(.,.) : R \times R \to R$ defined by $D_1(x,y) = D(a,x,y)$ and a mapping $d_2 : R \to R$ defined by $d_2(x) = D(a,a,x)$ is a symmetric bi-derivation (in this meaning, permuting 2-derivation is a symmetric bi-derivation) and is a derivation, respectively. If the symmetric bi-derivation and the derivation are obtained after some operations, studying at the permuting tri-derivation is not necessary.

THEOREM 7. *Let R be a non-commutative prime ring which is 2, 3-torsion free. Let $D(.,.,.)$ be a permuting tri-derivation of R and the trace of $D(.,.,.)$. If $[d(x), x] = 0$ for all $x \in R$, then $D = 0$.*

PROOF. From the hypothesis, for any $x, y \in R$

$$[d(x+y), x+y] + [d(-x+y), -x+y] = 0$$

and since R is 2, 3-torsion free we have, for all $x, y \in R$

$$[D(x,y,y), x] + [D(x,x,y), y] = 0. \tag{1}$$

Writing $y+z$, $z \in R$ for y in (1), from (1) we get, for all $x, y, z \in R$

$$2[D(x,y,z), x] + [D(x,x,y), z] + [D(x,x,z), y] = 0. \tag{2}$$

Replacing y by xy in (2) and from (2) we get, for all $x, y, z \in R$

$$2D(x,x,z)[y,x] + [x,z]D(x,x,y) + d(x)[y,z] = 0. \tag{3}$$

Replacing z by x in (3) and since R is 3-torsion free we obtain, for all $x, y \in R$

$$d(x)[y,x] = 0. \tag{4}$$

From (4) and Lemma 1 one can conclude that, for any $x \in Z$ we have $d(x) = 0$ (note that for any fixed $x \in R$ a mapping $y \mapsto [x,y]$ is a derivation). Let $x \in Z$ and $y \in Z$. Then, $x+y \in Z$, $-y \in Z$ and $x + (-y) \in Z$. Thus, $0 = d(x+y) = d(x) + 3D(x,y,y) + 3D(x,x,y)$ and $0 = d(x+(-y)) = d(x) + 3D(x,y,y) - 3D(x,x,y)$ which implies that

$$d(x) + 3D(x,y,y) = 0. \tag{5}$$

Writing $x+y$ for y in (5), from (5) and since R is 3-torsion free we get

$$d(x) + 2D(x,x,y) = 0. \tag{6}$$

Writing $-x$ for x in (6) we get,

$$-d(x) + 2D(x,x,y) = 0. \tag{7}$$

From (6) and (7) and since R is 2-torsion free we obtain $D(x,x,y) = 0$ Let us write in this relation $x+y$ instead of y we have $d(x) = 0$. Thus, we obtain that $d(x) = 0$ for all $x \in R$. From Lemma 5 we have $D = 0$.

THEOREM 8. *Let R be a non-commutative prime ring of characteristic not 2 and 3-torsion free. Let $D(.,.,.)$ be a permuting tri-derivation of R and d the trace of $D(.,.,.)$. If $[d(x),\ x] \in Z$ for all $x \in R$, then $D = 0$.*

PROOF. From the hypothesis, for any $x,\ y \in R$

$$[d(x+y), x+y] + [d(-x+y), -x+y] \in Z$$

and since $Char R \neq 2$ and R is 3-torsion free we have, for all $x,\ y \in R$

$$[D(x,y,y), x] + [D(x,x,y), y] \in Z. \tag{8}$$

Writing $y+z,\ z \in R$ for y in (8), from (8) we get, for all $x,\ y,\ z \in R$

$$(9) \qquad 2[D(x,y,z),x] + [D(x,x,y),z] + [D(x,x,z),y] \in Z.$$

Replacing x^2 by y in (9) we have, for all $x,\ z \in R$

$$(10) \qquad \begin{aligned} 3x[D(x,x,z),x] + 3[D(x,x,z),x]x + x[d(x),z] + [d(x),z]x \\ + [x,z]d(x) + d(x)[x,z] \in Z. \end{aligned}$$

Replacing x by z in (10), From the hypothesis and since $CharR \neq 2$ we have, for all $x \in R$

$$(11) \qquad x[d(x),x] \in Z.$$

Thus, we obtain, for all $x,y \in R$

$$(12) \qquad [x,y][d(x),x] = 0.$$

by (11) and the hypothesis. In this case, the relation above makes it possible to conclude, using the same arguments as the proof of Theorem 7, that for any $x \notin Z$ we have $[d(x),x] = 0$. Thus, from Theorem 7 we obtain $D = 0$.

THEOREM 9. *Let R be a prime ring of characteristic not 2 and 3-torsion free. Let $D_1(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R, d_1 and d_2, the traces of $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively. If $D_1(d_2(x),x,x) = 0$ for all $x \in R$, then $D_1 = 0$ or $D_2 = 0$.*

PROOF. For any $x,y \in R$

$$D_1(d_2(x+y),x+y,x+y) + D_1(d_2(-x+y),-x+y,-x+y) = 0$$

and since $CharR \neq 2$ we have, for all $x,y \in R$

$$(13) \qquad \begin{aligned} 2D_1(d_2(x),x,y) + D_1(d_2(y),x,x) + 6D_1(D_2(x,y,y),x,y) \\ + 3D_1(D_2(x,x,y),x,x) + 3D_1(D_2(x,x,y),y,y) = 0. \end{aligned}$$

Writing $y+z, z \in R$ for y in (13), from (13) and since R is 3-torsion free, we get

$$(14) D_1(D_2(y,z,z),x,x) + D_1(D_2(y,y,z),x,x) + 4D_1(D_2(x,y,z),x,y) +2D_1(D_2(x,y,y),x,z) + 4D_1(D_2(x,y,z),x,z) + 2D_1(D_2(x,z,z),x,y) +2D_1(D_2(x,y,y),x,z) + D_1(D_2(x,x,y),z,z) + D_1(D_2(x,x,z),y,y) +2D_1(D_2(x,x,z),y,z) = 0,$$

for all $x,y,z \in R$ Writing $-y$ for y in (14) we get, for all $x,y,z \in R$

$$(15) \quad - D_1(D_2(y,z,z),x,x) + D_1(D_2(y,y,z),x,x) + 4D_1(D_2(x,y,z),x,y) + 2D_1(D_2(x,y,y),x,z) - 4D_1(D_2(x,y,z),x,z) - 2D_1(D_2(x,z,z),x,y) + 2D_1(D_2(x,x,y),y,z) - D_1(D_2(x,x,y),z,z) - D_1(D_2(x,x,z),y,y) - 2D_1(D_2(x,x,z),y,z) = 0.$$

From (14) and (15) and since $Char R \neq 2$ we get, for all $x,y,z \in R$

$$(16) \quad D_1(D_2(y,y,z),x,x) + 2D_1(D_2(x,y,y),x,z) + 4D_1(D_2(x,y,z),x,y) + 2D_1(D_2(x,x,y),y,z) + D_1(D_2(x,x,z),y,y) = 0.$$

Replacing yz by z in (16), and from (16) we have, for all $x, y, z in R$

$$(17) \; D_1(x,x,y)D_2(y,y,z) + d_2(y)D_1(x,x,z) + 4D_1(x,y,y)D_2(x,y,z) +4D_2(x,y,y)D_1(x,y,z) + D_2(x,x,y)D_1(y,y,z) + d_1(y)D_2(x,x,z) = 0.$$

Replacing x by y in (17), since $Char R \neq 2$ and R is 3-torsion free we have, for all $x,y,z \in R$

$$(18) \qquad d_1(x)D_2(x,x,z) + d_2(x)D_1(x,x,z) = 0.$$

Writing yz for z in (18) and from (18) we obtain, for all $x,y,z \in R$

$$(19) \qquad d_1(x)yD_2(x,x,z) + d_2(x)yD_1(x,x,z) = 0.$$

Writing x for z in (19) and we have, for all $x, y \in R$

$$d_1(x)yd_2(x) = 0 \tag{20}$$

by Lemma 2. In this case, suppose that d_1 and d_2 are both different from zero. Then there exist $x_1, x_2 \in R$ such that $d_1(x_1) \neq 0$ and $d_2(x_2) \neq 0$. In particular, $d_1(x_1)yd_2(x_1) = 0$ for all $y \in R$. Since $d_1(x1) \neq 0$ and R is a prime ring we have $d_2(x_1) = 0$. Similarly, we get $d_1(x_2) = 0$. Then the relation (19) reduces to the equation $d_1(x_1)yD_2(x_1, x_1, z) = 0$ for all $y \in R$. Using this relation and Lemma 1 we obtain that $D_2(x_1, x_1, z) = 0$ for all $z \in R$ because of $d_1(x_1) \neq 0$ (recall that the mapping $z \mapsto D_2(x_1, x_1, z)$ is a derivation).Thus, we have $D_2(x_1, x_1, x_2) = 0$. In the same way, we get $D_1(x_1, x_1, x_2) = 0$. Substituting $x_1 + x_2$ for z, we obtain

$$\begin{aligned} d_1(z) &= d_1(x_1 + x_2) \\ &= d_1(x_1) + d_1(x_2) + 3D_1(x_1, x_1, x_2) + 3D_1(x_1, x_2, x_2) \\ &= d_1(x_1) \neq 0 \\ \text{and} \quad d_2(z) &= d_2(x_1 + x_2) \\ &= d_2(x_1) + d_2(x_2) + 3D_2(x_1, x_1, x_2) + 3D_2(x_1, x_2, x_2) \\ &= d_2(x_2) \neq 0. \end{aligned}$$

Therefore we have $d_1(z) \neq 0$ and $d_2(z) \neq 0$, a contradiction by (20) and R is prime ring. Hence we get $d_1(x) = 0$ for all $x \in R$ or $d_2(x) = 0$ for all $x \in R$. Thus, we have that $D_1 = 0$ or $D_2 = 0$ by Lemma 5.

COROLLARY 10. *Let R be a semi-prime ring of characteristic not 2 and 3-torsion free. Let $D(.,.,.)$ be a permuting tri-derivation of R and d be the trace of $D(.,.,.)$. If $D(d(x), x, x) = 0$ for all $x \in R$, then $D = 0$.*

PROOF. Replacing $D_1(.,.,.)$ and $D_2(.,.,.)$ by $D(.,.,.)$ in (20) we get that $d(x)yd(x) = 0$ for all $x, y \in R$. Thus, since R is a semi-prime ring we have $D = 0$ by Lemma 5.

THEOREM 11. *Let R be a prime ring of characteristic not 2 and 3, 5-torsion free. Let $D_1(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R, d_1 and d_2 be the traces of $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively. If $D_1(d_2(x), d_2(x), x) = 0$ for all $x \in R$, then $D_1 = 0$ or $D_1 = 0$*

PROOF. For any $x, y \in R$

$$D_1(d_2(x+y), d_2(x+y), x+y) + D_1(d_2(-x+y), d_2(-x+y), -x+y) = 0$$

and since $Char R \neq 2$ we have, for all $x, y \in R$

$$\begin{aligned}(21)\quad & 2D_1(d_2(y), d_2(x), x) + 6D_1(D_2(x,x,y), d_2(x), x) \\ & + 6D_1(D_2(x,y,y), d_2(y), x) + 18D_1(D_2(x,x,y), D_2(x,y,y), x) \\ & + D_1(d_2(x), d_2(x), y) + 6D_1(D_2(x,y,y), d_2(x), y) \\ & + 6D_1(D_2(x,x,y), d_2(y), y) + 9D_1(D_2(x,y,y), D_2(x,y,y), y) \\ & + 9D_1(D_2(x,x,y), D_2(x,x,y), y) = 0\end{aligned}$$

Writing $y + z, z \in R$ for y in (21), from (21) we get, for all $x, y, z \in R$

$$\begin{aligned}(22)\quad & 6D_1(D_2(y,z,z), d_2(x), x) + 2D_1(D_2(y,y,z), d_2(x), x) \\ & + 2D_1(D_2(x,y,y), d_2(z), x) + 2D_1(D_2(x,y,y), d_2(z), x) \\ & + 6D_1(D_2(x,y,y), D_2(y,y,z), x) + 18D_1(D_2(x,y,y), D_2(y,z,z), x) \\ & + 4D_1(D_2(x,y,z), d_2(y), x) + 12D_1(D_2(x,y,z), d_2(z), x) \\ & + 12D_1(D_2(x,y,z), D_2(y,z,z), x) + 36D_1(D_2(x,y,z), D_2(y,y,z), x) \\ & + 6D_1(D_2(x,z,z), d_2(y), x) + 6D_1(D_2(x,z,z), D_2(y,y,z), x) \\ & + 18D_1(D_2(x,z,z)D_2(y,z,z), x) + 12D_1(D_2(x,x,y), D_2(x,y,z), x) \\ & + 18D_1(D_2(x,x,y), D_2(x,z,z), x) + 6D_1(D_2(x,x,z), D_2(x,y,y), x) \\ & + 36D_1(D_2(x,x,z), D_2(x,y,z), x) + 2D_1(D_2(x,y,y), d_2(x), z) \\ & + 4D_1(D_2(x,y,z), d(x), y) + 12D_1(D_2(x,y,z), d_2(x), z) \\ & + 6D_1(D_2(x,z,z), d_2(x), y) + 2D_1(D_2(x,x,y), d_2(y), z) \\ & + 2D_1(D_2(x,x,y), d_2(z), y) + 6D_1(D_2(x,x,y), d_2(z), z) \\ & + 18D_1(D_2(x,x,y), D_2(y,z,z), y) + 6D_1(D_2(x,x,y), D_2(y,z,z), z) \\ & + 6D_1(D_2(x,x,y), D_2(y,y,z), y) + 18D_2(x,x,y), D_2(y,y,z), z)\end{aligned}$$

$$
\begin{aligned}
&+ 2D_1(D_2(x,x,z),d_2(y),y) + 6D_1(D_2(x,x,z),d_2(y),z)\\
&+ 6D_1(D_2(x,x,z),d_2(z),y) + 6D_1(D_2(x,x,z),D_2(y,z,z),y)\\
&+ 18D_1(D_2(x,x,z),D_2(y,z,z),z) + 18D_1(D_2(x,x,z),D_2(y,y,z),y)\\
&+ 6D_1(D_2(x,x,z),D_2(y,y,z),z) + 3D_1(D_2(x,x,y),D_2(x,x,y),z)\\
&+ 3D_1(D_2,(x,x,y),D_2(x,x,z),y) + 9D_1(D_2(x,x,y),D_2(x,x,z),z)\\
&+ 3D_1(D_2(x,x,z),D_2(x,x,y),y) + 9D_1(D_2(x,x,z),D_2(x,x,y),z)\\
&+ 9D_1(D_2(x,x,z),D_2(x,x,z),y) + 3D_1(D_2(x,y,y),D_2(x,y,y),z)\\
&+ 6D_1(D_2(x,y,y),D_2(x,y,z),y) + 18D_1(D_2(x,y,y),D_2(x,y,z),z)\\
&+ 9D_1(D_2(x,y,y),D_2(x,z,z),y) + 3D_1(D_2(x,y,y),D_2(x,z,z),z)\\
&+ 6D_1(D_2(x,y,y),D_2(x,y,z),y) + 18D_1(D_2(x,y,z),D_2(x,y,y),z)\\
&+ 36D_1(D_2(x,y,z),D_2(x,y,z),y) + 12D_1(D_2(x,y,z),D_2(x,y,z),z)\\
&+ 6D_1(D_2(x,y,z),D_2(x,z,z),y) + 18D_1(D_2(x,y,z),D_2(x,z,z),z)\\
&+ 9D_1(D_2(x,z,z),D_2(x,y,y),y) + 3D_1(D_2(x,z,z),D_2(x,y,y),z)\\
&+ 6D_1(D_2(x,z,z),D_2(x,y,z),y) + 18D_1(D_2(x,z,z),D_2(x,y,z),z)\\
&+ 9D_1(D_2(x,z,z),D_2(x,z,z),y) = 0.
\end{aligned}
$$

Thus, writing $-y$ for y in (22), from the equation is obtained and (22) and since $Char R \neq 2$ and R is 3-torsion free we get, for all $x,y,z \in R$

$$
\begin{aligned}
(23)\quad & 2D_1(D_2(y,y,z),d_2(x),x) + 2D_1(D_2(x,y,y),d_2(z),x)\\
&+4D_1(D_2(x,y,z),d_2(y),x) + 6D_1(D_2(x,y,y),D_2(y,y,z),x)\\
&+6D_1(D_2(x,z,z),D_2(y,y,z),x) + 12D_1(D_2(x,y,z),D_2(y,z,z),x)\\
&+12D_1(D_2(x,x,y),D_2(x,y,z),x) + 6D_1(D_2(x,x,z),D_2(x,y,y),x)\\
&+2D_1(D_2(x,y,y),d_2(x),z) + 4D_1(D_2(x,y,z),d_2(x),y)\\
&+2D_1(D_2(x,x,y),d_2(y),z) + 2D_1(D_2(x,x,y),d_2(z),y)\\
&+6D_1(D_2(x,x,y),D_2(y,z,z),z) + 6D_1(D_2(x,x,y),D_2(y,y,z),y)\\
&+2D_1(D_2(x,x,z),d_2(y),y) + 6D_1(D_2(x,x,z),D_2(y,z,z),y)\\
&+6D_1(D_2(x,x,z),D_2(y,y,z),z) + 3D_1(D_2(x,x,y),D_2(x,x,y),z)\\
&+3D_1(D_2(x,x,y),D_2(x,x,z),y) + 3D_1(D_2(x,x,z),D_2(x,x,y),y)
\end{aligned}
$$

$$+3D_1(D_2(x,y,y),D_2(x,y,y),z)+12D_1(D_2(x,y,y),D_2(x,y,z),y)$$
$$+3D_1(D_2(x,y,y),D_2(x,z,z),z)+12D_1(D_2(x,y,z),D_2(x,y,z),z)$$
$$+6D_1(D_2(x,y,z),D_2(x,z,z),y)+3D_1(D_2(x,z,z),D_2(x,y,y),z)$$
$$+6D_1(D_2(x,z,z),D_2(x,y,z),y)=0.$$

Writing $z+w, w \in R$ for z in (23) and from (23) we obtain, for all $x,y,z,w \in R$

$$\begin{aligned}
(24)\quad & 6D_1(D_2(x,y,y),D_2(z,w,w),x)+6D_1(D_2(x,y,y),D_2(z,z,w),x) \\
& +6D_1(D_2(x,z,z),D_2(y,y,w),x)+12D_1(D_2(x,z,w),D_2(y,y,z),x) \\
& +12D_1(D_2(x,z,w),D_2(y,y,w),x)+6D_1(D_2(x,w,w),D_2(y,y,z),x) \\
& +24D_1(D_2(x,y,z),D_2(y,z,w),x)+12D_1(D_2(x,y,z),D_2(y,w,w),x) \\
& +12D_1(D_2(x,y,w),D_2(y,z,z),x)+24D_1(D_2(x,y,w),D_2(y,z,w),x) \\
& +6D_1(D_2(x,x,y),D_2(z,w,w),y)+6D_1(D_2(x,x,y),D_2(z,z,w),y) \\
& +6D_1(D_2(x,x,y),D_2(y,z,z),w)+12D_1(D_2(x,x,y),D_2(y,z,w),z) \\
& +12D_1(D_2(x,x,y),D_2(y,z,w),w)+6D_1(D_2(x,x,y),D_2(y,w,w),z) \\
& +12D_1(D_2(x,x,z),D_2(y,z,w),y)+6D_1(D_2(x,x,z),D_2(y,w,w),y) \\
& +6D_1(D_2(x,x,w),D_2(y,z,z),y)+12D_1(D_2(x,x,w),D_2(y,w,z),y) \\
& +6D_1(D_2(x,x,z),D_2(y,y,z),w)+6D_1(D_2(x,x,z),D_2(y,y,w),z) \\
& +6D_1(D_2(x,x,z),D_2(y,y,w),w)+6D_1(D_2(x,x,w),D_2(y,y,w),z) \\
& +6D_1(D_2(x,x,w),D_2(y,y,z),z)+6D_1(D_2(x,x,w),D_2(y,y,z),w) \\
& +12D_1(D_2(x,y,y),D_2(x,z,z),w)+6D_1(D_2(x,y,y),D_2(x,z,w),z) \\
& +6D_1(D_2(x,y,y),D_2(x,z,w),w)+3D_1(D_2(x,y,y),D_2(x,w,w),z) \\
& +12D_1(D_2(x,y,z),D_2(x,y,z),w)+12D_1(D_2(x,y,z),D_2(x,y,w),z) \\
& +12D_1(D_2(x,y,z),D_2(x,y,w),w)+12D_1(D_2(x,y,w),D_2(x,y,z),z) \\
& +12D_1(D_2(x,y,w),D_2(x,y,z),w)+12D_1(D_2(x,y,w),D_2(x,y,w),z) \\
& +12D_1(D_2(x,y,z),D_2(x,z,w),y)+6D_1(D_2(x,y,z),D_2(x,w,w),y) \\
& +6D_1(D_2(x,y,w),D_2(x,z,z),y)+12D_1(D_2(x,y,w),D_2(x,z,w),y) \\
& +3D_1(D_2(x,z,z),D_2(x,y,y),w)+6D_1(D_2(x,z,w),D_2(x,y,y),z) \\
& +6D_1(D_2(x,z,w),D_2(x,y,y),w)+3D_1(D_2(x,w,w),D_2(x,y,y),z)
\end{aligned}$$

$$+ 6D_1(D_2(x,z,z), D_2(x,y,w), y) + 12D_1(D_2(x,z,w), D_2(x,y,z), y)$$
$$+ 12D_1(D_2(x,z,w), D_2(x,y,w), y) + 6D_1(D_2(x,z,z), D_2(x,y,z), y) = 0.$$

Replacing $-z$ by z in (24), from the equation is obtained and (24) and since $CharR \neq 2$ and R is 3-torsion free we get, for all $x, y, z, w \in R$

$$(25) \quad D_1(D_2(x,y,y), D_2(z,z,w), x) + D_1(D_2(x,z,z), D_2(y,y,w), x)$$
$$+2D_1(D_2(x,z,w), D_2(y,y,z), x) + 4D_1(D_2(x,y,z), D_2(y,z,w), x)$$
$$+2D_1(D_2(x,y,w), D_2(y,z,z), x) + D_1(D_2(x,x,y), D_2(z,z,w), y)$$
$$+D_1(D_2(x,x,y), D_2(y,z,z), w) + 2D_1(D_2(x,x,y), D_2(y,z,w), z)$$
$$+2D_1(D_2(x,x,z), D_2(y,z,w), y) + D_1(D_2(x,x,w), D_2(y,z,z), y)$$
$$+D_1(D_2(x,x,z), D_2(y,y,z), w) + D_1(D_2(x,x,z), D_2(y,y,w), z)$$
$$+D_1(D_2(x,x,w), D_2(y,y,z), z) + D_1(D_2(x,y,y), D_2(x,z,z), w)$$
$$+2D_1(D_2(x,y,z), D_2(x,y,z), w) + 2D_1(D_2(x,y,z), D_2(x,y,w), z)$$
$$+2D_1(D_2(x,y,w), D_2(x,y,z), z) + 2D_1(D_2(x,y,z), D_2(x,z,w), y)$$
$$+D_1(D_2(x,y,w), D_2(x,z,z), y) + D_1(D_2(x,z,w), D_2(x,y,y), z)$$
$$+D_1(D_2(x,z,z), D_2(x,y,w), y) + 2D_1(D_2(x,z,w), D_2(x,y,z), y) = 0.$$

Replacing x by y and z in (25), from the hypothesis we get, for all $x, w \in R$

$$(26) \qquad 5D_1(d_2(x), d_2(x), w) + 30D_1(D_2(x,x,w), d_2(x), x) = 0$$

Replacing wx by w in (26), from the hypothesis and (26) and since $CharR \neq 2$ and R is 3-torsion free we get, for all $x, w \in R$

$$(27) \qquad D_1(d_2(x), x, w)d_2(x) + D_2(x,x,w)D_1(d_2(x), x, x) = 0$$

Writing xw for w in (27) and from (27) we obtain, for all $x, w \in R$

$$(28) \qquad D_1(d_2(x), x, x)wd_2(x) + d_2(x)wD_1(d_2(x), x, x) = 0.$$

By Lemma 2 and from (28) we wish to get $D_1(d_2(x), x, x) = 0$ is fulfilled for all $x \in R$. But, if $D_1(d_2(x_1), x_1, x_1) \neq 0$ for some $x_1 \in R$ then replacing x by x_1 in (28) and since R is prime ring we obtain that $d_2(x_1) = 0$ by Lemma 2. Therefore, $D_1(d_2(x_1), x_1, x_1) = D_1(0, x_1, x_1) = 0$. But this contradicts to the fact that $D_1(d_2(x_1), x_1, x_1) \neq 0$. Thus, from Theorem 8 we get that $D_1 = 0$ or $D_2 = 0$.

COROLLARY 12. *Let R be a semi-prime ring of characteristic not 2 and 3, 5-torsion free. Let $D(.,.,.)$ be a permuting tri-derivation of R and d be the trace of $D(.,.,.)$. If $D(d(x),d(x),x) = 0$ for all $x \in R$, then $D = 0$.*

PROOF. Replacing $D_1(.,.,.)$ and $D_2(.,.,.)$ by $D(.,.,.)$ in (27) we get, for all $x, w \in R$. Thus, since R is semi-prime ring we have $D = 0$ by Lemma 5.

$$D(d(x),x,w)d(x) + D(x,x,w)D(d(x),x,x) = 0. \tag{29}$$

Replacing wy by y in (29) and from (29) we obtain for all $x, w \in R$

$$D(d(x),x,w)yd(x) + D(x,x,w)yD(d(x),x,x) = 0. \tag{30}$$

Replacing $d(x)$ by w in (30) and from the hypothesis we obtain for all $x, w \in R$

$$D(d(x),x,x)yD(d(x),x,x) = 0.$$

Thus, since R is semi-prime ring we have $D = 0$ by Corollary 10.

THEOREM 13. *Let R be a prime ring of characteristic not 2, 3 and 5. Let $D_2(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R, d_1 and d_2, the traces of $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively If*

$$d_1(d_2(x)) = f(x) \text{ for all } x \in R$$

then, $D_1 = 0$ or $D_2 = 0$, where a permuting tri-additive mapping $F(.,.,.) : R \times R \times R \to R$ and f is the trace of $F(.,.,.)$.

PROOF. Using the same argument as the proof of Theorem 11 we obtain that for all $x, y \in R$

$$D_1(D_2(x,y,y),d_2(y),d_2(y)) = 0. \tag{31}$$

Writing xy for x in (31) and from (31) we obtain, for all $x, y \in R$

$$D_2(x,y,y)D_1(y,d_2(y),d_2(y)) + D_1(x,d_2(y),d_2(y))d_2(y) = 0. \tag{32}$$

Writing yx for x in (32) and from (32) we obtain, for all $x, y \in R$

$$(33) \qquad d_2(y)xD_1(y, d_2(y), d_2(y)) + D_1(y, d_2(y), d_2(y))xd_2(y) = 0.$$

By Lemma 2 and from (33) we wish to get $D_1(d_2(y), d_2(y), y) = 0$ is fulfilled for all $y \in R$. But, if $D_1(d_2(y_1), y_1, y_1) \neq 0$ for some $y_1 \in R$ then replacing y by y_1 in (33) and since R is prime ring we obtain that $d_2(y_1) = 0$ by Lemma 2. Therefore, $D_1(d_2(y_1), d_2(y_1), y_1) = D_1(0, 0, x_1) = 0$. But this contradicts with $D_1(d_2(y_1), y_1, y_1) \neq 0$. Thus from Theorem 11 we get that $D_1 = 0$ or $D_2 = 0$.

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