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A NOTE ON MORLEY'S FORMULA

Young Joon Cho, Inhyok Park,
Tae Young Seo and Junesang Choi

Abstract Morley provided an interesting identity about 20 years earlier before its more generalized form was given by Dixon In this note some of its generalized forms and an application of Morley's formula are considered

1. Introduction and preliminaries

In 1902, Morley [10] established an interesting identity:

$$1+\sum_{n=1}^{\infty}\left\{\frac{a(a+1)\cdots(a+n-1)}{n!}\right\}^3=\cos\left(\frac{1}{2}\pi a\right)\frac{\Gamma\left(1-\frac{3}{2}a\right)}{\left\{\Gamma\left(1-\frac{1}{2}a\right)\right\}^3}, \tag{1.1}$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is

$$\Gamma(z)=\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}, \tag{1.2}$$

γ being the Euler-Mascheroni's constant defined by

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\cong 0.577\,215\,664\,901\,532\cdots \tag{1.3}$$

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If we make use of the Pochhammer symbol (or the shifted factorial) defined by

$$(1.4)\quad (\alpha)_n := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbf{N} := \{1,2,3,\cdots\}) \\ 1 & (n=0) \end{cases}$$

which, using the fundamental property of the Gamma function $\Gamma(z+1) = z\Gamma(z)$, is rewritten in the form:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{1.5}$$

and the generalized hypergeometric function notation ${}_pF_q$, we write the left member of (1.1) in the following form

$$\begin{aligned} 1+\sum_{n=1}^{\infty}\left\{\frac{a(a+1)\cdots(a+n-1)}{n!}\right\}^3 &= 1+\sum_{n=1}^{\infty}\left(\frac{(a)_n}{n!}\right)^3 \\ &= {}_3F_2\left[\begin{matrix} a,\ a,\ a\,; \\ 1,\ 1\,; \end{matrix}\ 1\right]. \end{aligned} \tag{1.6}$$

In 1922, Dixon evaluated a more general form of (1.6):

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma\,; \\ 1+\alpha-\beta,\ 1+\alpha-\gamma, \end{matrix}\ 1\right] \\ &= \frac{\Gamma\left(1+\frac{1}{2}\alpha\right)\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma\left(1+\frac{1}{2}\alpha-\beta-\gamma\right)}{\Gamma(1+\alpha)\Gamma\left(1+\frac{1}{2}\alpha-\beta\right)\Gamma\left(1+\frac{1}{2}\alpha-\gamma\right)\Gamma(1+\alpha-\beta-\gamma)}, \end{aligned} \tag{1.7}$$

which, for $\alpha = \beta = \gamma = a$, yields

$${}_3F_2\left[\begin{matrix} a,\ a,\ a\,; \\ 1,\ 1\,; \end{matrix}\ 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1-\frac{3}{2}a\right)}{\Gamma(1+a)\Gamma(1-a)\left\{\Gamma\left(1-\frac{1}{2}a\right)\right\}^2}, \tag{1.8}$$

which is easily seen to correspond with the right member of (1.1) by using the following well-known identity:

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}. \tag{1.9}$$

In this note we consider some more generalized forms of (1.6) and show how these kinds of summation formulas can be applied to evaluate certain infinite series.

Now we introduce the Psi (or Digamma) function is defined as the logarithmic derivative of the Gamma function:

$$\psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{1.10}$$

We give some well-known properties of the ψ–function (see [7]): For a positive integer n,

$$\begin{aligned} &\psi(1) = -\gamma, \qquad \psi(1/2) = -\gamma - 2\log 2, \\ &\psi(z+n) - \psi(z) = \sum_{k=0}^{n-1} \frac{1}{z+k}. \end{aligned} \tag{1.11}$$

The Polygamma functions are defined by (see [7, p. 41])

$$\psi^{(n)}(z) := \begin{cases} \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n \psi(z)}{dz^n} & (n \in \mathbf{N}) \\ \psi(z) & (n = 0) \end{cases} \tag{1.12}$$

from which it is easy to show that
(1.13)

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, z) \ \ (n \in \mathbf{N}),$$

where $\zeta(z, a)$ is the generalized (or Hurwitz) zeta function defined by

$$\zeta(z, a) = \sum_{k=0}^{\infty} (k+a)^{-z} \ \ (\mathrm{Re}(z) > 1;\ a \neq 0, -1, -2, \ldots) \tag{1.14}$$

and $\zeta(z, 1) = \zeta(z)$ is the Riemann zeta function. It is not difficult to derive the following results (see [15, pp. 265-275]):

$$\begin{aligned} &\zeta(z) = \frac{1}{1-2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^z} = \frac{1}{2^z - 1} \zeta\left(z, \frac{1}{2}\right) \ \ (\mathrm{Re}(z) > 1), \\ &\zeta(z, a) = \zeta(z, n+a) + \sum_{k=0}^{n-1} (k+a)^{-z} \ \ (n \in \mathbf{N}). \end{aligned} \tag{1.15}$$

2. Generalized forms of (1.6)

Recall the formula [14, p. 251].

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} a,\, b,\, c,\, d,\, e\,; \\ 1+a-b,\, 1+a-c,\, 1+a-d,\, 1+a-e\,; \end{matrix}\; 1\right] \\
&= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)} \\
&\quad \times {}_4F_3\left[\begin{matrix} 1+\frac{1}{2}a-b,\, c,\, d,\, e\,; \\ 1+\frac{1}{2}a,\, c+d+e-a,\, 1+a-b\,; \end{matrix}\; 1\right],
\end{aligned} \tag{2.1}
$$

which is subject to the restriction that one of the parameters $1+\frac{1}{2}a-b$, c, d, or e is a negative integer.

Taking the limit as $b \to \infty$ in (2.1) with the help of the asymptotic formula

$$
\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1+O\left(\frac{1}{z}\right)\right] \quad (z \to \infty;\ |\arg z| < \pi), \tag{2.2}
$$

we obtain

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, c,\, d,\, e\,; \\ 1+a-c,\, 1+a-d,\, 1+a-e\,; \end{matrix}\; 1\right] \\
&= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)} \\
&\quad \times {}_3F_2\left[\begin{matrix} c,\, d,\, e\,; \\ 1+\frac{1}{2}a,\, c+d+e-a\,; \end{matrix}\; 1\right]
\end{aligned} \tag{2.3}
$$

If we take the limit as $e \to \infty$ in (2.3) with the aid of (2.2) and use Gauss summation formula

$$
{}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \tag{2.4}
$$

we obtain Dixon's formula (1.7).

Exchanging b and e, and then writing $e = \frac{1}{2}(1 + a)$, we obtain a transformation formula for ${}_4F_3$.

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, b,\, c,\, d\,; \\ 1+a-b,\, 1+a-c,\, 1+a-d\,, \end{matrix}\; 1\right] \\
(2.5)\qquad &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \\
&\times {}_4F_3\left[\begin{matrix} \frac{1}{2},\, b,\, c,\, d\,, \\ 1+\frac{1}{2}a,\, \frac{1}{2}+\frac{1}{2}a,\, b+c+d-a; \end{matrix}\; 1\right],
\end{aligned}$$

which is subject to the restriction that one of the parameters b, c, or d is a negative integer

3. Infinite series

Many infinite series have been evaluated in terms of the Psi functions or the Zeta functions (see [8] and [9]).

Al-Saqabi *et al* [1] presented a systematic account of several interesting infinite series expressed in terms of the Psi (or Digamma) functions. Aular de Durán *et al.* [2] examined rather systematically the sums of numerous interesting families of infinite series with or without the use of fractional calculus.

Shen [12] investigated the connections between the Stirling numbers $s(n, k)$ of the first kind and the Riemann zeta function $\zeta(n)$ by means of the Gauss summation formula for ${}_2F_1$. In this line, various classes of infinite series have been evaluated by making use of known summation formulas for ${}_2F_1$ and ${}_3F_2$ (see [3], [4], and [5])

In this section we also evaluate certain infinite series by using the formula (1.8).

For our purpose we introduce the Stirling numbers $s(n, k)$ of the first kind defined by the following equation (see [6, pp. 204-218], [11,

p. 43], and [13, p. 396, Problem 25]):

$$z(z-1)\cdots(z-n+1)=\sum_{k=0}^{n} s(n,k)z^k.$$

From the above definition of $s(n,k)$, the Pochhammer symbol (or the shifted factorial) can be written in the form:

$$(z)_n = z(z+1)\cdots(z+n-1)=\sum_{k=0}^{n}(-1)^{n+k}s(n,k)z^k \tag{3.1}$$

It is also not difficult to see that

$$\begin{aligned}
&(-1)^{n+1}s(n,1)=(n-1)!; \qquad (-1)^n s(n,2)=(n-1)!\sum_{k=1}^{n-1}\frac{1}{k};\\
&(-1)^{n+1}s(n,3)=\frac{(n-1)!}{2}\left\{\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)^2-\sum_{k=1}^{n-1}\frac{1}{k^2}\right\}.
\end{aligned} \tag{3.2}$$

Replace a by z in (1 8) and set

$$\begin{aligned}
f(z):&=1+\sum_{n=1}^{\infty}\left(\frac{(z)_n}{n!}\right)^3\\
&=\frac{\Gamma\left(1+\frac{1}{2}z\right)\Gamma\left(1-\frac{3}{2}z\right)}{\Gamma(1+z)\,\Gamma(1-z)\left\{\Gamma\left(1-\frac{1}{2}z\right)\right\}^2}\\
&=:\sum_{n=0}^{\infty}a_n\,z^n.
\end{aligned} \tag{A}$$

From (3.1),

$$(z)_n=\sum_{k=0}^{n}A_k z^k, \tag{B}$$

where

$$A_k = (-1)^{n+k} s(n,k).$$

We therefore have

$$f(z) = 1 + \sum_{n=1}^{\infty} \left(\alpha_3 z^3 + \alpha_4 z^4 + \alpha_5 z^5 + \cdots \right), \tag{C}$$

where

$$\begin{aligned}
\alpha_3 &= \frac{A_1^3}{(n!)^3} = \frac{1}{n^3}, \\
\alpha_4 &= \frac{3A_1^2 A_2}{(n!)^3} = \frac{3}{n^3} \left(\sum_{k=1}^{n-1} \frac{1}{k} \right), \\
\alpha_5 &= \frac{3A_1 \left(A_1 A_3 + A_2^2 \right)}{(n!)^3} = \frac{3}{2n^3} \left[3 \left(\sum_{k=1}^{n-1} \frac{1}{k} \right)^2 - \sum_{k=1}^{n-1} \frac{1}{k^2} \right].
\end{aligned}$$

Applying the logarithmic derivative to (A) leads immediately to

$$\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{1}{2}\psi\left(1 + \frac{1}{2}z\right) - \frac{3}{2}\psi\left(1 - \frac{3}{2}z\right) \\
&\quad - \psi(1+z) + \psi(1-z) + \psi\left(1 - \frac{1}{2}z\right).
\end{aligned} \tag{D}$$

The application of (1.13) to (D) yields

$$\frac{f'(z)}{f(z)} := \sum_{n=0}^{\infty} c_n z^n, \tag{E}$$

where $c_0 = 0$ and

$$c_n = \left[\frac{(-1)^{n+1} - 2}{2^{n+1}} + (-1)^n - 1 + \left(\frac{3}{2}\right)^{n+1} \right] \zeta(n+1) \quad (n \in \mathbf{N}). \tag{F}$$

From (A) and (E), we have

$$\begin{aligned} f'(z) &= \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n \\ &= \frac{f'(z)}{f(z)} \cdot f(z) = \left(\sum_{n=0}^{\infty} c_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k}c_k\right) z^n, \end{aligned}$$

which, upon equating the coefficients of z^n, yields

$$(n+1)a_{n+1} = \sum_{k=0}^{n} a_{n-k}\, c_k \quad (n \in \mathbf{N}). \tag{G}$$

Note also that

$$a_1 = \frac{f'(0)}{1!} = \frac{f'(0)}{f(0)} = c_0. \tag{H}$$

Now, using (F) to (H), we obtain $a_2 = 0$, and

$$a_3 = \zeta(3), \quad a_4 = \frac{3}{4}\zeta(4), \quad a_5 = \frac{3}{2}\zeta(5), \dots . \tag{I}$$

Finally, from (C) and (I), we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right) = \frac{5}{4}\zeta(4), \tag{3.3}$$

which was evaluated in an elementary way by using the formula (1.6) (see Morley [10, p. 402, Eq.(6)]);

$$\sum_{n=1}^{\infty} \frac{1}{n^3}\left[3\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)^2 - \sum_{k=1}^{n-1}\frac{1}{k^2}\right] = \zeta(5). \tag{3.4}$$

Or, equivalently, (3.4) is rewritten in the form

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left[3\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2-\sum_{k=1}^{n}\frac{1}{k^2}\right]=6\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\sum_{k=1}^{n}\frac{1}{k}\right)-3\zeta(5). \tag{3.5}$$

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References

[1] B. N Al-Saqabi, S L Kalla, and H. M Srivastava, *A certain family of infinite series associated with Digamma functions*, J Math Anal Appl **159** (1991), 361–372

[2] J. Aular de Durán, S L Kalla, and H M Srivastava, *Fractional calculus and the sums of certain families of infinite series*, J. Math. Anal Appl. **190** (1995), 738–754

[3] J. Choi and T Y Seo, *Evaluation of some infinite series*, Indian J Pure Appl. Math **28** (1997), 791–796

[4] J Choi and H M Srivastava, *Certain classes of infinite series*, Monatsh Math. **127** (1999), 15–25

[5] J Choi, H M Srivastava, T Y Seo, and A K Rathie, *Some families of infinite series*, Soochow J Math **25** (1999), 209–219

[6] L Comtet, *Advanced Combinatorics The Art of Finite and Infinite Expansions* (Translated from the French by J W. Nienhuys), Reidel, Dordrecht and Boston, 1974

[7] H T Davis, *The Summation of Series*, The Principia Press of Trinity University, San Antonio, Texas, 1962

[8] I S Gradshteyn and I M Ryzhik, *Tables of Integrals, Series and Products* (Corrected and Enlarged Ed prepared by A Jeffrey), Academic Press, 1980.

[9] E R Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

[10] F Morley, *On the series* $1+\left(\frac{p}{1}\right)^3+\left\{\frac{p(p+1)}{1\,2}\right\}^3+$, Proc London Math Soc.(1) **34** (1902), 397–402

[11] G. Pólya and G Szego, *Problems and Theorems in Analysis* Translated from the German by D Aeppli), Vol I, Springer-Verlag, New York, Heidelberg and Berlin, 1972

[12] Li-Chien Shen, *Remarks on some integrals and series involving the Stirling numbers and $\zeta(n)$*, Trans Amer Math Soc. **347** (1995), 1391–1399

[13] H M Srivastava and H L Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984

[14] F. J. W Whipple, *On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum*, Proc. London Math. Soc.(2) **24** (1926), 247–263

[15] E T. Whittaker and G. N Watson, *A Course of Modern Analysis,* Fourth Ed., Cambridge University Press, Cambridge, London, and New York, 1958

Young Joon Cho, Inhyok Park and Tae Young Seo
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: tyseo@hyowon.pusan.ac.kr

Junesang Choi
Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail: junesang@email.dongguk.ac.kr