East Asian Math J 15(1999), No 2, pp. 223-232

THE MAXIMUM DETERMINANT OF (0,1)-TRIDIAGONAL MATRICES

Geum-sug Hwang

ABSTRACT In this paper, we give the upper bound of determinants of $(0,1)$-tridiagonal matrices and we show that the $(0,1)$-tridiagonal matrices which have maximal determinant are sign-nonsingular

1. Introduction and preliminaries

In 1993, Li Ching[2] showed that the *Lower Hessenberg* $n \times n$ $(0,1)$-matrix have maximal determinant $F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$, $F_1 = F_2 = 1$, where F_n is the nth Fibonacci number He proved that the matrix $H_n = [h_{ij}]$ of order n defined by

$$h_{i(i-k)} = \begin{cases} 1 & \text{if } k \in \{-1, 0, 2, 4, . \quad | i-k > 0\} \\ 0 & \text{otherwise} \end{cases}$$

has maximal determinant F_n.

Let n be a positive integer, $n \geq 2$. An $n \times n$ $(0,1)$- matrix $A = [a_{ij}]$ is said to be a tridiagonal matrix if $a_{ij} = 0$ for $|i - j| > 1$. There are c_n(possibly nonzero) terms in the determinant of a tridiagonal matrix of order n where $c_n = c_{n-1} + c_{n-2}, c_2 = 2, c_3 = 3$, i e. $c_n = F_{n+1}$. So this is a trivial upper bound for the determinant

The definition of a sign-nonsingular $(0,1)$-matrix is given in [1] and we now give a well known theorem about sign-nonsingular matrices.

Received April 11, 1999 Revised June 12, 1999.

THEOREM 1[1]. *Let $A_n = [a_{ij}]$ be a matrix of order n. Then the following are equivalent.*

i) *A_n is a sign-nonsingular matrix;*
ii) *$Det\ A_n \neq 0$ and A_n has signed determinant;*
iii) *There is a nonzero term in the standard determinant expansion of A_n and every nonzero term has the same sign.*

We found that the matrix H_n above has the property that $det\ H_n = per\ H_n$, where $per\ H_n$ is the permanent of H_n. This means that there is no cancellation in the nonzero terms in determinant expansion. Hence H_n is a sign-nonsingular matrix by Theorem 1.

In this paper we investigate the upper bound of absolute values of determinants of (0, 1)-tridiagonal matrices and we want to show that (0, 1)-tridiagonal matrices which have maximal determinant are the sign-nonsingular matrices.

For an $n \times n$-matrix A_n, we define that $A_n[i_1, i_2, \ldots, i_k]$ is the submatrix obtained from A_n by deleting all rows and columns not in $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$.

Let $A_n = [a_{ij}]$ be an $n \times n$ (0, 1)-tridiagonal matrix. Then

$$\begin{aligned} det\ A_n &= a_{11} \cdot det\ A_n[2, \ldots, n] - a_{12}a_{21} \cdot det\ A_n[3, \ldots, n] \\ &= det\ A_n[1,2] \cdot det\ A_n[3, \ldots, n] - a_{11} \cdot a_{23}a_{32} \cdot det\ A_n[4, \ldots, n] \\ &= det\ A_n[1,2,3] \cdot det\ A_n[4, \ldots, n] - det\ A_n[1,2] \cdot a_{34}a_{43} \\ &\qquad \times det\ A_n[5, \ldots, n] \\ &= \cdots \\ &= det\ A_n[1, \ldots, n-2] \cdot det\ A_n[n-1, n] - det\ A_n[1, \ldots, n-3] \\ &\qquad \times a_{n-2\ n-1}a_{n-1\ n-2}a_{nn}. \end{aligned}$$

Therefore we have

$$\begin{aligned} (1) \qquad det\ A_n &= det\ A_n[1, \ldots, i] \cdot det\ A_n[i+1, \ldots, n] \\ &\quad - det\ A_n[1, \ldots, i-1] \cdot a_{ii+1}a_{i+1i} \cdot det\ A_n[i+2, \ldots, n] \end{aligned}$$

for any i, $1 \leq i \leq n-2$ where $det\ A_n[1, k] = 1$ for $k < 1$.

LEMMA 2. *Let $A_n = [a_{ij}]$ be an $n \times n$ $(0,1)$-tridiagonal matrix such that $a_{kk+1} = 0$ or $a_{k+1k} = 0$ for some k, $1 \le k \le n-1$. Then*

$$det\ A_n = det\ A_n[1, \dots, k] \cdot det\ A_n[k+1, \dots, n]$$

PROOF. Without loss of generality, assume that $a_{kk+1} = 0$. Since $a_{ij} = 0$ for all i, j such that $|i-j| > 1$, $a_{ij} = 0$ for all i, $1 \le i \le k$ and j, $k+1 \le j \le n$. Thus A_n has of the form $A_n = \begin{bmatrix} A_n[1, \dots, k] & 0 \\ * & A_n[k+1, \dots, n] \end{bmatrix}$ and so we get the lemma.

From now on, we denote $A_n = A_n[1, \dots, k] \oplus A_n[k+1, \dots, n]$ if $a_{kk+1} = 0$ or $a_{k+1k} = 0$ and let $A \oplus B = B \oplus A$ since the determinants of them are equal.

LEMMA 3. *Let $A_n = [a_{ij}]$ be an $n \times n$ $(0,1)$-tridiagonal matrix such that $a_{ij} = 1$ for $|i-j| \le 1$. Then*

$$det\ A_n = \begin{cases} (-1)^k & \text{if } n = 3k \text{ or } n = 3k+1 \\ 0 & \text{if } n = 3k+2 \end{cases}$$

PROOF. Let $A_n = [a_{ij}]$ be a $(0,1)$-tridiagonal matrix such that $a_{ij} = 1$ for all i and j and let $n = 3k + l$, $l = 0, 1$ and 2. Use induction on k. Since $a_{ii} = 1$ for all i, from (1)

$$\begin{aligned} detA_n &= det\ A_n[2, 3, \dots, n] - det\ A_n[3, 4, \dots, n] \\ &= det\ A_n[3, 4, \dots, n] - det\ A_n[4, 5, \dots, n] - det\ A_n[3, 4, \dots, n] \qquad (2) \\ &= (-1) \cdot det\ A_n[4, 5, \dots, n]. \end{aligned}$$

For $k = 1$, $det\ A_3 = det\ A_3[2,3] - 1 = 0 - 1 = -1$, $det\ A_4 = det\ A_4[2,3,4] - det\ A_4[3,4] = -1 - 0 = -1$ And $det\ A_5 = (-1) \cdot det\ A_5[4,5] = 0$. Assume that $k \ge 2$. From (2), we have $det\ A_n = (-1) \cdot det\ A_n[4, 5, \dots, n]$. Since $A_n[4, 5, \dots, n]$ is an $(n-3) \times (n-3)$ $(0,1)-$ tridiagonal matrix, by inductive hypothesis,

$$det\ A_n[4, 5, \dots, n] = \begin{cases} (-1)^{k-1} & \text{if } n = 3k \text{ or } n = 3k+1 \\ 0 & \text{if } n = 3k+2. \end{cases}$$

Hence $det\ A_n = -1 \cdot (-1)^{k-1} = (-1)^k$ for $n = 3k$ or $n = 3k+1$ and $det\ A_n = 0$ for $n = 3k+2$.

2. Main results

THEOREM 4. *For $n = 2k+1$, define a tridiagonal matrix D_n by*

$$D_n = [d_{ij}] = \begin{cases} 0 & \text{if } i = j \ (i = \text{even}) \text{ or } |i-j| > 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then $\det D_n = (-1)^k \cdot (k+1)$ and there are $k+1$ nonzero terms in the standard expansion of determinant such that every nonzero term has the same sign.

PROOF. Use induction on k. For $k = 1$, $\det D_3 = -d_{11}d_{23}d_{32} - d_{12}d_{21}d_{33} = (-1)\cdot 2$, and there are two nonzero terms which have the same sign. Assume that this result is true for all $r < k$ and let $n = 2k+1$. Then by (1), since $a_{22} = a_{44} = \cdots = a_{n-1\,n-1} = 0$,

$$\begin{aligned} \det D_n &= \det D_n[2,\ldots,n] - \det D_n[3,\ldots,n] \\ &= (-1)\det D_n[4,\ldots,n] - \det D_{n-2} \\ &= \cdots = (-1)^{k-1}\det D_n[n-1,n] - \det D_{n-2} \\ &= (-1)^k - \det D_{n-2} \end{aligned}$$

since $D_n[3,\ldots,n] = D_{n-2}$ and $\det D_n[n-1,n] = -1$. By inductive hypothesis, there are k nonzero terms in the determinant expansion of D_{n-2} and $\det D_{n-2} = (-1)^{k-1}\cdot k$. Hence $\det D_n = (-1)^k - (-1)^{k-1}\cdot k = (-1)^k\cdot(k+1)$ and this implies that there are $k+1$ nonzero terms and so every nonzero term has the same sign

COROLLARY 5. *The matrices D_n defined above are sign-nonsingular if $n = 2k+1$.*

THEOREM 6. *Let $A_n = [a_{ij}]$ be an $n\times n$ $(0,1)$- tridiagonal matrix. Then*

$$|\det A_n| \le \begin{cases} 2 & \text{for } n = 3,\ 4 \\ 3 & \text{for } n = 5. \end{cases}$$

Furthermore the equality holds if and only if

$$A_n = \begin{cases} D_3 & \text{for } n = 3 \\ D_3 \oplus I_1 & \text{for } n = 4 \\ D_5 & \text{for } n = 5. \end{cases}$$

PROOF. For $n = 3$, $det\ A_3 = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$. So $|\ det\ A_3\ | \leq 2$ and equality holds if and only if $a_{22} = 0$ and $a_{11} = a_{12} = a_{21} = a_{23} = a_{32} = a_{33} = 1$. Therefore the equality holds if and only if $A_3 = D_3$.

For $n = 4$, $det\ A_4 = det\ A_4[1, 2] \cdot det\ A_4[3, 4] - a_{11}a_{23}a_{32}a_{44}$ from (1). Since for any 2×2 $(0, 1)-$ matrix A_2, $|\ det\ A_2\ | = 1$ or 0, $|\ det\ A_4| \leq 2$. And the equality holds if and only if $a_{11} = a_{23} = a_{32} = a_{44} = 1$ and $det\ A_4[1, 2] \cdot det\ A_4[3, 4] = -1$. Therefore $|\ det\ A_4\ | = 2$ if and only if $A_4 = D_3 \oplus I_1$ where I_1 is identity matrix of order 1.

For $n = 5$, $det\ A_5 = det A_5[1, 2, 3] \cdot det\ A_5[4, 5] - det\ A_5[1, 2] \cdot a_{34}a_{43} \cdot a_{55}$ by (1). If $A_5[1, 2, 3] \neq D_3$ or one of a_{34}, a_{43}, a_{55} equals to 0, then $|\ det\ A_5\ | \leq 2$ Assume that $A_5[1, 2, 3] = D_3$ and $a_{34} = a_{43} = a_{55} = 1$. Then $|\ det\ A_5\ | = | - 2 \cdot (a_{44} - a_{45}a_{54}) + 1| \leq 3$ and the equality holds if and only if $a_{44} = 0$ and $a_{45} = a_{54} = 1$. Thus $det\ A_5 = 3$ if and only if $A_5 = D_5$

REMARK. *Let $A_n = [a_{ij}]$ be an $n \times n$ $(0, 1)$-tridiagonal matrix such that $a_{ij} = 1$ for all i, j with $|i - j| = 1$. Let s be the smallest positive integer such that $a_{ss} = 0$. Then $det\ A_n = -det\ A_n[1, \ldots, s-2] \cdot det\ A_n[s+1, \ldots, n] - det\ A_n[1, \ldots, s-1] \cdot det\ A_n[s+2, \ldots, n]$ by determinant cofactor expansion along the sth row. Therefore, for some l,*

$$(3) \qquad det\ A_n = \begin{cases} (-1)^{l-1} \cdot det\ A_n[s+1, \ldots, n] & if\ \ s = 3l \\ (-1)^{l-1} \cdot det\ A_n[s+2, \ldots, n] & if\ \ s = 3l+1 \\ (-1)^{l} \cdot [det\ A_n[s+1, \ldots, n] & \\ \qquad + det\ A_n[s+2, \ldots, n]] & if\ \ s = 3l+2 \end{cases}$$

since $det\ A_n[1, \ldots, s-2] = 0$ or $det\ A_n[1, \ldots, s-1] = 0$ for $s \neq 3l+2$ by Lemma 3.

THEOREM 7. *Let $A_n = [a_{ij}]$ be an $n \times n$ $(0, 1)$-tridiagonal matrix. Then*

$$|\ det\ A_n\ | \leq \begin{cases} 2^2 & for\ \ n = 6,\ 7 \\ 3 \cdot 2 & for\ \ n = 8. \end{cases}$$

Furthermore the equality holds if and only if

$$\begin{cases} A_6 = D_3 \oplus D_3 \\ A_7 = D_3 \oplus D_3 \oplus I_1 \ \textit{or}\ A_7 = D_7 \\ A_8 = D_5 \oplus D_3. \end{cases}$$

PROOF. Case 1: Assume that $a_{ii+1} = a_{i+1i} = 1$ for all i, $1 \le i \le n-1$. From (1),

$$det\ A_n = det\ A_n[1,2,3] \cdot det\ A_n[4,\dots,n] - det\ A_n[1,2] \cdot det\ A_n[5,\dots,n].$$

For $n = 6$ and 7, if $A_n[1,2,3] \neq D_3$ or $A_n[n-2,n-1,n] \neq D_3$ then $|\ det\ A_n\ | \le 3 < 4$. Assume that $A_n[1,2,3] = A_n[n-2,n-1,n] = D_3$. Then $detA_6 = (-2)(-2) - (-1)(-1) = 3 < 4$ and $detA_7 = -2\{a_{44} \cdot (-2) + 1\} - 2 = 4 \cdot a_{44} - 4$ So $|\ detA_7\ | \le 4$ and the equality holds if and only if $A_7 = D_7$.

For $n = 8$, if $A_8[1,2,3] \neq D_3$ (this implies $|\ det\ A_8[1,2,3]\ | \le 1$) or if $A_8[4,\dots,8] \neq D_5$(this implies $|\ det\ A_8[4,\dots,8]\ | \le 2$), then $|\ det\ A_8\ | \le 5$ since $A_8[5,6,7,8] \neq D_3 \oplus I_1$. Assume that $A_8[1,2,3] = D_3$ and $A_8[4,\dots,8] = D_5$. Then $|\ det\ A_8\ | = |\ (-2)\cdot 3 + 1\ | = 5 < 6$.

Case 2: Assume that $a_{ii+1} = 0$ or $a_{i+1i} = 0$ for some i. Without loss of generality, assume that $1 \le i \le \lceil \frac{n}{2} \rceil$. By Lemma 2, $detA_n = detA_n[1,\dots,i] \cdot detA_n[i+1,\dots,n]$. Then $|\ det\ A_6\ | \le 3, 2, 4$ for $i = 1,2,3$ respectively and $|\ det\ A_7\ | \le 4,3,4,4$ for $i = 1,2,3,4$ respectively. So $|\ det\ A_n\ | \le 2^2$ for $n = 6,7$ and the equality holds if and only if $A_6 = D_3 \oplus D_3$ and $A_7 = I_1 \oplus (D_3 \oplus D_3)$ for $i = 1$, $D_3 \oplus (D_3 \oplus I_1)$ for $i = 3$ and $(D_3 \oplus I_1) \oplus D_3$ for $i = 4$. Thus $|\ det\ A_7\ | = 4$ if $A_7 = D_3 \oplus D_3 \oplus I_1$. For $n = 8$, $|\ det\ A_8\ | \le 4, 4, 2\cdot 3, 4$ for $i = 1,2,3,4$ respectively. Thus $|\ det\ A_8\ | \le 2 \cdot 3$ and the equality holds if and only if $A_8 = D_3 \oplus D_5 = D_5 \oplus D_3$.

THEOREM 8. *Let $A_n = [a_{ij}]$ be an $n \times n$ $(0,1)$-tridiagonal matrix. Then*

$$|\ det\ A_n\ | \le \begin{cases} 2^3 & \textit{for } n = 9 \\ 3^2 & \textit{for } n = 10 \\ 3 \cdot 2^2 & \textit{for } n = 11. \end{cases}$$

Furthermore the equality holds if and only if $\begin{cases} A_9 = & D_3 \oplus D_3 \oplus D_3 \\ A_{10} = & D_5 \oplus D_5 \\ A_{11} = & D_5 \oplus D_3 \oplus D_3. \end{cases}$

PROOF. Case 1: Suppose that $a_{ii+1} = a_{i+1i} = 1$ for all i, $1 \leq i \leq n-1$. If $a_{11} = 0$ or $a_{11} = a_{22} = 1$ then

$$| \, det\ A_n \, | \leq \begin{cases} 4 & \text{for } n = 9 \\ 6 & \text{for } n = 10 \end{cases}$$

by (1). Assume that $a_{11} = 1$ and $a_{22} = 0$. Then

$$det\ A_n = -\{\, det\ A_n[3, \ldots, n] + det\ A_n[4, \ldots, n]\}.$$

For $n = 9$, $| \, det\ A_9[3, \ldots, 9] \, | \leq 4$ and $| \, det\ A_9[4, \ldots, 9] \, | \leq 3$ by case 1 in Theorem 7 Thus $| \, det\ A_9 \, | \leq 7 < 8$. For $n = 10$, $| \, det\ A_{10}[3, \ldots, 10] \, | \leq 5$, $| \, det\ A_{10}[4, \ldots, 10] \, | \leq 4$.

But if $| \, det\ A_{10}[4, \ldots, 10] \, | = 4$ then $A_{10}[4, \ldots, 10] = D_7$ and so $| \, det\ A_{10}[3, \ldots, 10] \, | \leq 3$. Thus $| \, det\ A_{10} \, | \leq 8 < 9$. For $n = 11$, if $a_{11} = 0$ or $a_{11} = a_{22} = 1$ then $| \, det\ A_{11} \, | \leq 7 < 12$ and so assume that $a_{11} = 1$, $a_{22} = 0$. If $a_{33} = 0$ or $a_{33} = a_{44} = 1$ then $| \, det\ A_{11} \, | \leq 9 < 12$ Assume that $a_{33} = 1$, $a_{44} = 0$. Then $| \, det\ A_{11} \, | = | \, det\ A_{11}[5, \ldots, 11] + det\ A_{11}[6, \ldots, 11] \, | \leq 10 < 12$.

Case 2: Suppose that $a_{ii+1} = 0$ or $a_{i+1i} = 0$ for some i, $1 \leq i \leq n-1$. Then $det\ A_n = det\ A_n[1, \ldots, i] \cdot det\ A_n[i+1, \ldots, n]$ by Lemma 2. By Theorem 6 and 7,

$$| \, det\ A_n \, | \leq \begin{cases} 2^3 & \text{for } n = 9 \\ 3^2 & \text{for } n = 10 \\ 3 \cdot 2^2 & \text{for } n = 11. \end{cases}$$

The equality holds if and only if $A_n = \begin{cases} D_3 \oplus D_3 \oplus D_3 & \text{for } n = 9 \\ D_5 \oplus D_5 & \text{for } n = 10 \\ D_5 \oplus D_3 \oplus D_3 & \text{for } n = 11. \end{cases}$

We now give the main result about the maximal determinant of (0, 1)-tridiagonal matrices.

THEOREM 9. *Let $A_n = [a_{ij}]$ be an $n \times n$ $(0,1)$-tridiagonal matrix for $n \geq 9$. Then*

$$| \, det \; A_n \, | \leq \begin{cases} 2^k & if \;\; n = 3k \\ 3^2 \cdot 2^{k-3} & if \;\; n = 3k+1 \\ 3 \cdot 2^{k-1} & if \;\; n = 3k+2. \end{cases}$$

Furthermore the equality holds if and only if

$$A_n = \begin{cases} D_3 \oplus \cdots \oplus D_3 & if \;\; n = 3k \\ D_5 \oplus D_5 \oplus D_3 \oplus \cdots \oplus D_3 & if \;\; n = 3k+1 \\ D_5 \oplus D_3 \oplus \cdots \oplus D_3 & if \;\; n = 3k+2 \end{cases}$$

PROOF. We use induction on k where $n = 3k+l$, $l = 0, 1$ or 2 $(k \geq 3)$. We proved this for $k = 3$ in Theorem 8. Assume that $k > 3$ and the theorem is true for $n \leq 3(k-1)+l$, $l = 0, 1$ or 2.

Case 1: Suppose that $a_{ii+1} = a_{i+1i} = 1$ for all i, $1 \leq i \leq n-1$. Since $a_{23} = a_{32} = 1$, $det \; A_n = det \; A_n[1,2,3] \cdot det \; A_n[4,\ldots,n] - det \; A_n[1,2] \cdot det \; A_n[5,\ldots,n]$ by (1). If $A_n[1,2,3] \neq D_3$ then $| \, det \; A_n[1,2,3] \, | \leq 1$ and $| \, det \; A_n[1,2] \, | \leq 1$. So we can say $| \, det \; A_n \, | \leq | \, det \; A_n[4,\ldots,n] \, | + | \, det \; A_n[5,\ldots,n] \, |$. Thus, by inductive hypothesis,

$$| \, det \; A_n \, | \leq \begin{cases} 2^{k-1} + 3 \cdot 2^{k-3} < 2^k & \text{for} \;\; l = 0 \\ 3^2 \cdot 2^{k-4} + 2^{k-1} < 3^2 \cdot 2^{k-3} & \text{for} \;\; l = 1 \\ 3 \cdot 2^{k-2} + 3^2 \cdot 2^{k-4} < 3 \cdot 2^{k-1} & \text{for} \;\; l = 2. \end{cases}$$

Assume that $A_n[1,2,3] = D_3$. If $a_{4,4} = 1$, then

$$det \; A_n = -det \; A_n[4,\ldots,n] + det \; A_n[6,\ldots,n].$$

Hence,

$$| \, det \; A_n \, | \leq \begin{cases} 2^{k-1} + 3^2 \cdot 2^{k-5} < 2^k & \text{for} \;\; l = 0 \\ 3^2 \cdot 2^{k-4} + 3 \cdot 2^{k-3} < 3^2 \cdot 2^{k-3} & \text{for} \;\; l = 1 \\ 3 \cdot 2^{k-2} + 2^{k-1} < 3 \cdot 2^{k-1} & \text{for} \;\; l = 2. \end{cases}$$

Let $a_{44} = 0$. Then $det\ A_n = det\ A_n[5, \ldots, n] + 2\ det\ A_n[6, \ldots, n]$. For $l = 0$, $|\ det\ A_n\ | \leq 2 \cdot 3^2 \cdot 2^{k-5} + 3 \cdot 2^{k-3} < 2^k$. For $l = 1, 2$, if $a_{55} = 0$ or $a_{55} = a_{66} = 1$ then $|\ det\ A_n\ | \leq 2 \cdot |\ det\ A_n[6, \ldots, n]\ | + |\ det\ A_n[7, \ldots, n]\ |$. So

$$|\ det\ A_n\ | \leq \begin{cases} 2 \cdot 3 \cdot 2^{k-3} + 3^2 \cdot 2^{k-5} < 3^2 \cdot 2^{k-3} & \text{for } l = 1 \\ 2 \cdot 2^{k-1} + 3 \cdot 2^{k-3} < 3 \cdot 2^{k-1} & \text{for } l = 2. \end{cases}$$

If $a_{55} = 1$ and $a_{66} = 0$ then $det\ A_n = det\ A_n[5, \ldots, n] - 2 \cdot det\ A_n[8, \ldots, n]$. Thus

$$|\ det\ A_n\ | \leq \begin{cases} 2^{k-1} + 2 \cdot 2^{k-2} < 3^2 \cdot 2^{k-3} & \text{for } l = 1 \\ 3^2 \cdot 2^{k-4} + 3^2 \cdot 2^{k-5} < 3 \cdot 2^{k-1} & \text{for } l = 2. \end{cases}$$

Case 2: Suppose that $a_{ii+1} = 0$ or $a_{i+1i} = 0$ for some i, $1 \leq i \leq n-1$. By Lemma 2, $det\ A_n = det\ A_n[1, \ldots, i] \cdot det\ A_n[i+1, \ldots, n]$. Let $i = 3s + t$ for some s, $t = 0, 1$ or 2 and we now apply inductive hypothesis to $A_n[1, \ldots, i]$ and $A_n[i+1, \ldots, n]$. For $l = 0$,

$$|\ det\ A_n\ | \leq \begin{cases} 2^s \cdot 2^{k-s} \leq 2^k & \text{for } t = 0 \\ 3^3 \cdot 2^{k-5} < 2^k & \text{for } t = 1 \text{ or } 2. \end{cases}$$

Furthermore, the equality holds if and only if $A_n = (D_3 \oplus \cdots \oplus D_3) \cdot (D_3 \oplus \cdots \oplus D_3)$ For $l = 1$,

$$|\ det\ A_n\ | \leq 3^2 \cdot 2^{k-3} \quad \text{for } t = 0, 1 \text{ or } 2$$

And the equality holds if and only if $A_n = (D_5 \oplus D_5 \oplus D_3 \oplus \cdots \oplus D_3) \cdot (D_3 \oplus \cdots \oplus D_3)$ or $A_n = (D_5 \oplus D_3 \oplus \cdots \oplus D_3) \cdot (D_5 \oplus D_3 \oplus \cdots \oplus D_3)$ Hence $|\ det\ A_n\ | = 3^2 \cdot 2^{k-3}$ if and only if $A_n = D_5 \oplus D_5 \oplus D_3 \oplus \cdots \oplus D_3$. For $l = 2$,

$$|\ det\ A_n\ | \leq \begin{cases} 3 \cdot 2^{k-1} & \text{for } t = 0 \text{ or } 2 \\ 3^4 \cdot 2^{k-6} < 3 \cdot 2^{k-1} & \text{for } t = 1. \end{cases}$$

Furthermore the equality holds if and only if $A_n = (D_3 \oplus \cdots \oplus D_3) \cdot (D_5 \oplus D_3 \oplus \cdots \oplus D_3)$ or $A_n = (D_5 \oplus D_3 \oplus \cdots \oplus D_3) \cdot (D_3 \oplus \cdots \oplus D_3)$ Hence $|\ det\ A_n\ | = 3 \cdot 2^{k-1}$ if and only if $A_n = D_5 \oplus D_3 \oplus \cdots \oplus D_3$.

COROLLARY 10. *The* $(0,1)$*-tridiagonal matrices* A_n, $n \geq 3$ *which has maximum determinant are sign-nonsingular.*

PROOF. The $(0,1)$-tridiagonal matrices A_n, $n \geq 3$, $n \neq 4, 7$ which has maximal determinant can be expressed as direct product of sign-nonsingular matrices D_3 and D_5 by theorem 6,7 and 9. The direct product of sign-nonsingular matrices is also sign-nonsingular. For $n =$ 4 and 7, it is easy to check that $D_3 \oplus I_1$ and $D_3 \oplus D_3 \oplus I_1$ are sign-nonsingular.

REFERENCES

[1] R. A. Brualdi and B L. Shader, *The Matrices of Sign-Solvable Linear Systems*, 1996

[2] Li Ching, *The Maximum Determinant of an* $n \times n$ *Lower Hessenberg* $(0,1)$*-matrix*, Linear Algebra and its Applications **183** (1993), 147-153.

Division of Applied Mathematics and Statistics
Pusan University of Foreign Studies
Pusan 608-701, Korea
E-mail: gshwang@taejo.pufs.ac.kr