East Asian Math J. 15(1999), No. 2, pp 247-253

ON STRONG REGULARITY AND STRONG REDUCIBILITY OF NEAR-RINGS

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1. Introduction

Throughout this paper N represents a (right) near-ring. N is called *zero symmetric* if $a0 = 0$ for all $a \in N$, *left s-unital* if $a \in Na$, *reduced* if it has no nonzero nilpotent element, *left bipotent* if $Na = Na^2$ for all $a \in N$. The definitions of regularity and π-regularity for near-rings are the same concepts as for rings.

In 1980, Mason[3] introduced the notions of left strong regularity, right strong regularity, left regularity and right regularity of near-rings. He proved that for a zero symmetric unital near-ring, the notions of left strong regularity, left regularity and right regularity are equivalent. Also in 1998, Mason[4] researched several properties on strong forms of regularity for near-rings In 1984, some properties of strong regularity have been slightly improved by Reddy and Murty[8], and also in 1986, that of strong regularity and strong π-regularity of semigroup were investigated by Hongan[2].

N is said to be *left strongly regular* if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$, that is, a is a left strongly regular element. N is called *left regular* if N is left strongly regular and regular. Right strong regularity and right regularity are defined in a symmetric way. In the ring theory, left strong regularity and right strong regularity are equivalent, but in near-ring theory, they are different in Mason[3]. So we say that left strongly regular and right strongly regular near-ring is *strongly regular.*

Received April 15, 1999 Revised June 7, 1999

More generally, a near-ring N is called *left strongly π-regular* if for all $a \in N$, there exists a positive integer n such that a^n is a left strongly regular element, *left π-regular* if it is left strongly π-regular and π-regular. In a symmetric way, *right strong π-regularity* and *right π-regularity* are defined.

First we investigate some characterizations of left strong regularity of near-rings, that is, the five equivalent concepts of left strong regurality, from this statement, for arbitrary ring case, we obtain 9-equivalent conditions of strong regularity.

Next, we introduce a special concept of near-ring, that is, strongly reduced property Cho and Hirano[1] in 1999, which is more improved than Reddy and Murty's (*) property[8], and investigate some properties of strong reducibility of near-rings.

The remainder concepts in near-ring theory can be found in Pilz[7].

2. Properties on strong regularity and strong reducibility of near-rings

LEMMA 2.1[3]. *Let N be a zero symmetric and reduced near-ring. Then for any $a, b \in N$, $ab = 0$ implies $ba = 0$ and N has the I.F.P.*

LEMMA 2.2[9]. *Let N be a left strongly regular near-ring. If $a = xa^2$ for some a, x in N, then $a = axa$ and $ax = xa$.*

THEOREM 2.3. *Let N be a zero symmetric near-ring. Then the following statements are equivalent:*

(1) N is left s-unital and left bipotent;

(2) N is reduced and left bipotent;

(3) N is left strongly regular;

(4) N is regular and for any $a \in N$ there exists $x \in N$ such that $ax = xa$;

(5) N is left regular.

PROOF. (1) $\Longrightarrow$ (2). Suppose N is left s-unital and left bipotent. Let a be a nilpotent element in N. Then there exists a positive integer n such that $a^n = 0$. Since N is left s-unital and left bipotent,

$$a \in Na = Na^2 = Na^3 = \cdots = Na^n = N0 = \{0\}.$$

Hence N is reduced.

(2) $\Longrightarrow$ (3). Assume N is reduced and left bipotent. Then for each $a \in N$,

$$Na = Na^2 = Na^3$$

So we have the equation $a^2 = xa^3$ for some x in N. This implies that $(a - xa^2)a = 0$ By Lemma 2.1, $a(a - xa^2) = 0$. Thus

$$(a - xa^2)^2 = a(a - xa^2) - xa^2(a - xa^2) = 0 - 0 = 0.$$

Since N is reduced, $a = xa^2$. Hence N is left strongly regular.

(3) $\Longrightarrow$ (4). This is proved from Lemma 2.2.

(4) $\Longrightarrow$ (1). Suppose the conditions (4). Let $a \in N$. Then there exists $x \in N$ such that $a = axa$ and $ax = xax$ Thus clearly, $a \in Na$ so that N is left s-unital. On the other hand, for any $ra \in Na$,

$$ra = raxa = rxa^2$$

which is contained in Na^2. Hence N is left bipotent.

(4) $\Longrightarrow$ (5) $\Longrightarrow$ (3) are easily proved.

LEMMA 2.4. *Every right s-unital and right bipotent near-ring is reduced.*

PROOF. Let N be a right s-unital and right bipotent near-ring. If a is a nilpotent element of N, then there exists a positive integer n such that $a^n = 0$. Since N is right s-unital and right bipotent,

$$a \in aN = a^2N = a^3N = \cdots = a^nN = 0N = 0.$$

Hence N is reduced.

For any ring case, It is well known that left strong regularity and right strong regularity are equivalent, so we have the following equivalent conditions of strong regularity from Theorem 2.3 and Lemma 2.4.

COROLLARY 2.5. *Let N be an arbitrary ring. Then the following statements are equivalent:*

(1) N is left s-unital and left bipotent;

(2) N is reduced and left bipotent;

(3) N is left strongly regular;

(4) N is regular and for any $a \in N$ there exists $x \in N$ such that $ax = xa$;

(5) N is left regular;

(6) N is right regular;

(7) N is right strongly regular;

(8) N is reduced and right bipotent;

(9) N is right s-unital and right bipotent;

(10) N is strongly regular.

Now we introduce the following special concept of near-rings(Cho and Hirano[1]). N is called a *strongly reduced near-ring* if it satisfied the following condition: For $a \in N$, $a^2 \in N_c$ implies $a \in N_c$, where $N_c = \{a|a0 = a\} = \{a|ab = a \quad \text{for all} \quad b \in N\}$ is the *zero-symmetric part* of N.

LEMMA 2.6.([1] PROPOSITION 1) *Let N be a strongly reduced near-ring. Then the following statements hold:*

(i) For any $a, b \in N$, $ab = 0$ implies $ba = b0$.

(ii) N is reduced.

(iii) If $e \in N$ is an idempotent, then $ene = en$ for all $n \in N$.

Thus every strongly reduced near-ring is reduced, but not conversely, there are many examples. In the ring theory, any idempotent e satisfying the condition of Lemma 2 6 (iii) is said to be *right semi-central.*

THEOREM 2.7.. *Let N be an arbitrary near-ring. Then the following statements are equivalent:*

(1) N is strongly reduced;

(2) For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

PROOF. (1) $\Longrightarrow$ (2). Assume that $a^3 = a^2$. Then $(a^2 - a)a = 0$. Then by Lemma 2.6(i) $a(a^2 - a) = a0 \in N_c$. Then $(a^2 - a)a^2 = (a^3 - a^2)a = 0a = 0 \in N_c$. Again by Lemma 2.6(i) $a^2(a^2 - a) = a^2 0 \in N_c$. Hence $(a^2 - a)^2 = a^2(a^2 - a) - a(a^2 - a) = a^2 0 - a0 = (a^2 - a)0 \in N_c$. This implies $a^2 - a \in N_c$. Hence $a^2 - a = (a^2 - a)0 = (a^2 - a)a = 0$.

(2) $\Longrightarrow$ (1). Assume $a^2 \in N_c$. Then $a^3 = a^2 a = a^2 0 a = a^2 0 = a^2$. From our assumption (2), this implies $a = a^2 \in N_c$.

By Lemma 2 6 and Theorem 2.7 we know that strongly reducibility is equivalent to the (*) property by Reddy and Murty[8]

EXAMPLES 2.8.

(1) Every right strongly regular near-ring N is strongly reduced, indeed, for each $a \in N$ with $a^3 = a^2$, we have $a = a^2 x$ for some $x \in N$ and $a^2 = a(a^2 x) = a^3 x = a^2 x$ so that $a^2 = a$. Hence by Theorem 2.7, N is strongly reduced.

Also, we have the following examples:

(2) Every left strongly regular near-ring is strongly reduced.

(3) Every integral near-ring is strongly reduced.

(4) Every constant near-ring is strongly reduced.

(5) $\mathbb{Z}_2$ and $\mathbb{Z}_2[x]$ are strongly reduced.

THEOREM 2.9. *Let N be a strongly reduced near-ring. If N is left strongly π-regular, then N is π-regular with the property that for every $a \in N, a^n x = x a^n$ for some positive integer n and $x \in N$*

PROOF. Suppose N is a left strongly π-regular near-ring. Then for any $a \in N$, there exists $x \in N$ such that $a^n = x a^{2n}$ for some positive integer n. From this equation, we have the following equation:

$$(a^n - a^n x a^n)a^n = 0.$$

From Lemma 2 6(i), we see that

$$a^n(a^n - a^n x a^n) = a^n 0, \quad a^n x a^n (a^n - a^n x a^n) = a^n x a^n 0.$$

On the other hand, $(a^n - a^n x a^n)^2 = a^n(a^n - a^n x a^n) - a^n x a^n (a^n - a^n x a^n) = a^n 0 - a^n x a^n 0$, also, $(a^n - a^n x a^n)^3 = (a^n 0 - a^n x a^n 0)(a^n -$

$a^nxa^n) = a^n0 - a^nxa^n0$. From Theorem 2.7,

$$(a^n - a^nxa^n)^2 = a^n - a^nxa^n.$$

Consequently, $0 = (a^n - a^nxa^n)a^n = (a^n - a^nxa^n)^2a^n = (a^n0 - a^nxa^n0)a^n = (a^n0 - a^nxa^n0) = (a^n - a^nxa^n)^2 = a^n - a^nxa^n$, namely, $a^n = a^nxa^n$. Hence N is π-regular.

Finally, from the equation $(a^nx - xa^n)a^n = 0$ and Lemma 2.6(i), we get $a^n(a^nx - xa^n) = a^n0$ and $a^nx(a^nx - xa^n) = a^nx0$. However

$$(a^nx - xa^n)^2 = a^nx(a^nx - xa^n) - xa^n(a^nx - xa^n) = a^nx0 - xa^n0$$

also,

$$(a^nx - xa^n)^3 = (a^nx - xa^n)^2(a^nx - xa^n) = (a^nx0 - xa^n0)(a^nx - xa^n).$$

By Theorem 2.7, $(a^nx - xa^n)^2 = a^nx - xa^n$. Consequently, we see that $0 = (a^nx - xa^n)a^n = (a^nx - xa^n)^2a^n = (a^nx0 - xa^n0) = a^nx - xa^n$. Therefore $a^nx = xa^n$.

Therefore N is strongly π-regular.

PROPOSITION 2.10. *Let N be a strongly reduced near-ring. If for $x, y \in N, xy = 0$, then xay is a constant element, for all $a \in N$.*

PROOF. Suppose N is a strongly reduced near-ring and for $x, y \in N, xy = 0$. Then $yx = y0$ and for each $a \in N$,

$$(xay)^2 = xayxay = xay0$$

Similarly, $(xay)^3 = xay0$. Since N is strongly reduced, $xay = (xay)^2$, that is, $xay = xay0$. Hence xay is a constant element for all $a \in N$.

From this Proposition 2.10, obviously every zero symmetric strongly reduced near-ring has the $I.F.P.$

COROLLARY 2.11. *Let N be near-ring with constant multiplication. Then N is strongly reduced near-ring if and only if $N = \{0\}$.*

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