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THE SPACES OF ULTRADIFFERENTIABLE FUNCTIONS OF TWO TYPES AND WHITNEY JETS

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0. Introduction

We investigate some problems on the space $E_M(\Omega)$(resp. $E_{[\omega]}(\Omega)$) of ultradifferentiable functions of class $M = (M_k)_{k\in N_0(=N\cup\{0\})}$ (resp. Beurling, Roumieu type) and that of $E_M(K)$(resp. $E_{[\omega]}(K)$) of Whitney jets of class M(resp. Beurling, Roumieu type) on a compact set K in R^n. Also we consider the problems on non quasi-analytic classes of functions. Here M(resp. $[\omega]$) stands for (M_k) or $\{M_k\}$(resp. (ω) or $\{\omega\}$).

1. The spaces $E_{[\omega]}(\Omega)$ and $E_M(\Omega)$

Throughout N_0 and N denote the sets of all nonnegative integers and positive integers, respectively. Let $M = (M_k)_{k\in N_0}$ be a sequence of positive numbers which satisfies some of the following conditions with $M_0 = 1$,

(M.1) $M_k^2 \le M_{k-1}M_{k+1},\ k \in N$;

(M.2) There are constants $K > 0$ and $H > 1$ such that

$$M_k \le KH^k \min_{0\le l\le k} M_l M_{k-l},\ k \in N_0;$$

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(M.3) There is a constant $L > 0$ such that

$$\sum_{l=k+1}^{\infty} \frac{M_{l-1}}{M_l} \leq Lk \frac{M_k}{M_{k+1}}, \quad k \in N;$$

(M.3)′
$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$$

We write $m_k = \dfrac{M_k}{M_{k-1}}, k \in N$, and define

$m(t) =$ the number of $m_k \leq t, M(t) = \sup_k \log \dfrac{t^k}{M_k}$.

DEFINITION 1.1. Let $w : [0, \infty) \to [0, \infty)$ be a continuous increasing function with $w(0) = 0$ and $\lim_{t\to\infty} w(t) = \infty$. We consider the following conditions on w:

(α) $0 = w(0) \leq w(s+t) \leq w(s) + w(t)$ for all s, $t \in [0, \infty)$,

(β) $\int_0^\infty \frac{w(t)}{1+t^2} dt < \infty$;

(γ) $\lim_{t\to\infty} \frac{\log t}{w(t)} = 0$;

(δ) $\varphi : t \to w(e^t)$ is convex;

(ϵ) There exists $C > 0$ with $\int_1^\infty \frac{w(yt)}{t^2} dt \leq Cw(y) + C$ for all $y \geq 0$;

(ζ) There exists $H \geq 1$ with $2w(t) \leq w(Ht) + H$ for all $t \geq 0$.

DEFINITION 1.2 A continuous increasing function $w : [0, \infty) \to [0, \infty)$ with $w(0) = 0$ will be called a *weight function* if it has the properties (γ), (δ) and (ϵ) in Definition 1.1.

We shall denote by $\varphi^* . [0, \infty) \to [0, \infty)$ the *Young conjugate* of the convex function φ defined in (δ), i e.,

$$\varphi^*(s) = \sup\{st - \varphi(t) | t \geq 0\}.$$

DEFINITION 1.3. Let Ω be an open set in R^n.

(a) We define the space $E_{(\omega)}(\Omega)$ (resp. $E_{\{\omega\}}(\Omega)$) of ultradifferentiable functions of *Beurling type* (resp. *Roumieu type*) is the set of C^∞-functions f in Ω with the property that for each compact set $K \subset \Omega$

and each $n \in N$(resp. some $n \in N$), $||f||_{K,n}$(resp. $||f||_{K,\frac{1}{n}}$) is finite, where

$$||f||_{K,n} = \sup_{\substack{x \in K \\ \alpha}} \frac{|\partial^\alpha f(x)|}{\exp[\varphi^*(\frac{|\alpha|}{n})]}.$$

The topology in the space $E_{(\omega)}(\Omega)$ (resp. $E_{\{\omega\}}(\Omega)$) is given the projective limit topology over K and n(resp. the projective limit topology over K of the inductive limit over n).

(b) We define the space $E_{(M_k)}(\Omega)$ (resp. $E_{\{M_k\}}(\Omega)$) of ultradifferentiable functions of class (M_k)(resp. $\{M_k\}$), i.e., $E_{(M_k)}(\Omega)$ (resp $E_{\{M_k\}}(\Omega)$)$= \{f \in C^\infty(\Omega)$: for each compact set $K \subset \Omega$ and each(resp. some) $h > 0$, $P_{K,h}(f) < \infty\}$, where

$$P_{K,h}(f) = \sup_{\substack{x \in K \\ \alpha}} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

The topology on the space $E_{(M_k)}(\Omega)$ (resp. $E_{\{M_k\}}(\Omega)$) is given the projective limit topology over K and $n \in N$(resp. the projective limit topology over K of the inductive limit over $n \in N$).

THEOREM 1.4. *For a compact set $K \subset \Omega$ and $n \in N$, we define*

$$(1.1) \qquad \begin{aligned} &E_{(\omega)}(K,n) = \{f \in C^\infty(\Omega) : ||f||_{K,n} < \infty\}, \\ &E_{\{\omega\}}(K,n) = \{f \in C^\infty(\Omega) . ||f||_{K,\frac{1}{n}} < \infty\}, \\ &E_{(M_k)}(K,n) = \{f \in C^\infty(\Omega) . P_{K,\frac{1}{n}} < \infty\}, \text{ and} \\ &E_{\{M_k\}}(K,n) = \{f \in C^\infty(\Omega) . P_{K,n} < \infty\}. \end{aligned}$$

Then

$$(1.2) \qquad \begin{aligned} &E_{(\omega)}(\Omega) = \text{proj} \lim_{\substack{K \Subset \Omega \\ n \to \infty}} E_{(\omega)}(K,n), \\ &E_{\{\omega\}}(\Omega) = \text{proj} \lim_{K \Subset \Omega} [\text{ind} \lim_{n \to \infty} E_{\{\omega\}}(K,n)], \\ &E_{(M_k)}(\Omega) = \text{proj} \lim_{\substack{K \Subset \Omega \\ n \to \infty}} E_{(M_k)}(K,n), \text{ and} \\ &E_{\{M_k\}}(\Omega) = \text{proj} \lim_{K \Subset \Omega} [\text{ind} \lim_{n \to \infty} E_{\{M_k\}}(K,n)]. \end{aligned}$$

PROOF. They are well defined and can be proved by the following remark :

(i)
$$\exp[n\varphi^*(\frac{|\alpha|}{n})] = \sup_{t\geq 0}[\frac{t^{|\alpha|}}{\exp nw(t)}],$$
$$\exp[\varphi^*(\frac{1}{n}|\alpha|)] = \sup_{t\geq 0}[\frac{t^{|\alpha|}}{\exp \frac{1}{n}w(t)}].$$

(ii)
$$\exp[nw(t)] = \sup_{x\geq 0}\{\frac{t^x}{\exp[n\varphi^*(\frac{x}{n})]}\},$$
$$\exp[\frac{1}{n}w(t)] = \sup_{x\geq 0}\{\frac{t^x}{\exp[\frac{1}{n}\varphi^*(nx)]}\}.$$

THEOREM 1.5. *The space $E_{\{M_k\}}(R^n)$ of ultradifferentiable functions of class $\{M_k\}$ is a Silva space, that is, inductive limit of Fréchet spaces such that the canonical mappings are compact.*

PROOF. We define, for $j \in N$, $E_{M,j}(R^n) = \{f \in C^\infty(R^n)$: for every compact set K in R^n, $P_{K,j}(f) < \infty\}$, where the topology in $E_{M,j}(R^n)$ is defined by, for an increasing sequence $\{K_i\}$ of compact sets such that $\cup K_i = R^n$, the system of seminorms $\{P_{K_i,j} : i \in N\}$.

Then $E_{M,j}(R^n)$ is a Fréchet space with $\{P_{K_i,j} : i \in N\}$. We have the well-known properties such that the image of a bounded set under a continuous linear mapping is a bounded set and the closure of a bounded set is bounded. Also the space $E_{M,j}(R^n)$ has the property that all bounded and closed subsets are compact by Ascoli's theorem. Therefore, for $j < k$, the inclusion mappings $E_{M,j}(R^n) \hookrightarrow E_{M,k}(R^n)$ are compact. Hence we have
$$E_{\{M_k\}}(R^n) = \text{ind} \lim_{j\to\infty} E_{M,j}(R^n).$$

THEOREM 1.6. *The space $E_{\{\omega\}}(R^n)$ of ultradifferentiable functions of Roumieu type is a Silva space*

Indeed, if a sequence$(M_k)_{k\in N_0}$ of positive numbers satisfies (M.1), (M.2) and (M.3), then there exists a weight function $w(t)$ satisfying all conditions (α)-(ζ) in Definition 1.1 such that $E_{\{M_k\}}(R^n) = E_{\{\omega\}}(R^n)$ and vice versa (see [6], Theorems 3.1, 3.2).

2. Non quasi-analyticity

(M.1) is equivalent to saying that the sequence

$$m_k = \frac{M_k}{M_{k-1}},\ k \in N \tag{2.1}$$

is increasing. We denote by $m(t)$ the number of $m_k \leq t$. Then we have

$$M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda. \tag{2.2}$$

PROPOSITION 2.1. *(1)* $\int_0^\infty \frac{M(t)}{t^2} dt < \infty \Rightarrow \lim_{t\to\infty} \frac{M(t)}{t} = 0$.
(2) $\int_0^\infty \frac{m(t)}{t^2} dt < \infty \Rightarrow \lim_{t\to\infty} \frac{m(t)}{t} = 0$.

PROOF (1) For otherwise, there exists a constant $\epsilon > 0$ and a sequence $t_1 < t_2 < \cdots$ such that $\frac{M(t_j)}{t_j} > \epsilon$ and $t_{j+1} > 2t_j$. Hence

$$\int_0^\infty \frac{M(t)}{t^2} dt \geq \sum_{j=1}^\infty \int_{t_j}^{2t_j} \frac{M(t_j)}{t^2} dt = \sum_{j=1}^\infty \frac{M(t_j)}{2t_j} = \infty$$

Hence $\lim_{t\to\infty} \frac{M(t)}{t} = 0$
(2) Similarly we can prove it

THEOREM 2.2. $\lim_{k\to\infty} \frac{k}{m_k} = 0 \Leftrightarrow \lim_{t\to\infty} \frac{m(t)}{t} = 0$.

PROOF ($\Rightarrow$) For every $\epsilon > 0$, there exists $n_0 \in N$ such that $\frac{k}{m_k} < \epsilon$ when $k \geq n_0$, i.e., $\frac{k}{\epsilon} < m_k$ for $k \geq n_0$. If $t = \frac{1}{\epsilon} n_0$, then $\frac{m(t)}{t} = \frac{m(\frac{1}{\epsilon} n_0)}{\frac{1}{\epsilon} n_0}$ $\leq \frac{n_0}{\frac{1}{\epsilon} n_0} = \epsilon$.

($\Leftarrow$) For every $\epsilon = \frac{1}{n}$, there exists $M > 0$ such that $t \geq M$ implies $\frac{m(t)}{t} < \frac{1}{n}$, i.e., there exists $n_0 \in N$ such that $t \geq nn_0$ implies $\frac{m(nn_0+k)}{nn_0+k} < \frac{1}{n}$. If $k = n_0$, then $\frac{k}{m_k} = \frac{n_0}{m_{n_0}} \leq \frac{n_0}{nn_0} = \frac{1}{n} \to 0$ as $n \to \infty$.

PROPOSITION 2.3. $\lim_{t\to\infty} \frac{m(t)}{t} = 0 \Leftrightarrow \lim_{t\to\infty} \frac{M(t)}{t} = 0$.

PROOF. Obvious by L'Hôpital's law.

PROPOSITION 2.4. ([7], Proposition 3.1) *Suppose that* $M = (M_k)_{k\in N_0}$ *satisfies (M.1) and (M.3)′ , then*

(1) $\displaystyle\int_0^\infty \frac{M(t)}{t^2}dt = \int_0^\infty \frac{m(t)}{t^2}dt,$

(2) $\displaystyle\int_0^\infty \frac{dm(t)}{t} = \int_0^\infty \frac{m(t)}{t^2}dt.$

PROOF. See [7].

PROPOSITION 2.5. ([7], Proposition 3.5) *Suppose that* $M = (M_k)_{k\in N_0}$ *satisfies (M.1) and (M.3)′. Then we have the following relations:*

(1) $\displaystyle\int_0^t \frac{m(\lambda)}{\lambda^2}d\lambda = \frac{M(t)}{t} + \int_0^t \frac{M(\lambda)}{\lambda^2}d\lambda,$

(2) $\displaystyle\int_t^\infty \frac{M(\lambda)}{\lambda^2}d\lambda = \frac{M(t)}{t} + \int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda,$ *and hence by (1) or (2) we have*

(3) $\displaystyle\int_0^\infty \frac{M(\lambda)}{\lambda^2}d\lambda = \int_0^\infty \frac{m(\lambda)}{\lambda^2}d\lambda.$ *By (1) and (2) we have*

(4) $\displaystyle\int_0^t \frac{m(\lambda)-M(\lambda)}{\lambda^2}d\lambda = \int_t^\infty \frac{M(\lambda)-m(\lambda)}{\lambda^2}d\lambda.$

PROOF. See [7].

Suppose that $M = (M_k)_{k\in N_0}$ satisfies (M.1). Then M satisfies (M.3) if and only if there is a constant A such that

$$(*) \qquad \int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda \le (A+1)\frac{m(t)}{t}$$

for $t \ge m_1$ (see Komatsu [4], Proposition 4.4).

Integrating both sides of (*), we obtain for $t \ge m_1$

$$\int_{m_1}^t d\mu \int_\mu^\infty \frac{m(\lambda)}{\lambda^2}d\lambda = t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda + \int_{m_1}^t \frac{m(\lambda)}{\lambda}d\lambda - m_1\int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2}d\lambda$$

$$\le (A+1)\int_{m_1}^t \frac{m(\mu)}{\mu}d\mu = (A+1)M(t).$$

Hence we have the following relation :

$$(1)\qquad t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda \le A\int_{m_1}^t \frac{m(\lambda)}{\lambda}d\lambda + m_1\int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2}d\lambda$$
$$= AM(t) + m_1\int_0^\infty \frac{m(\lambda)}{\lambda^2}d\lambda.$$

By Proposition 2.5(2), we have $t\int_t^\infty \frac{M(\lambda)}{\lambda^2}d\lambda = M(t)+t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda.$ Hence, by (1) and Proposition 2.5(3), we have the relation :

$$(2)\qquad t\int_t^\infty \frac{M(\lambda)}{\lambda^2}d\lambda \le (A+1)M(t) + m_1\int_0^\infty \frac{M(\lambda)}{\lambda^2}d\lambda.$$

3. Whitney jets of class M on K

The letters α, β will mean multi-indexes in N_0^n. For $\alpha = (\alpha_1, \cdots, \alpha_n)$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ Also, $\alpha \le \beta$ stands for $\alpha_i \le \beta_i (i = 1, \cdots, n)$ and, for $x \in R^n, x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let K be a compact set in R^n.

A *jet* in K is a multisequence $F = (f_\alpha)$ of continuous functions f_α on K. For a jet F, for $x, y \in K, z \in R^n, m \in N_0$ and $|\alpha| \le m$, we put

$$(3.1)\qquad (T_x^m F)(z) = \sum_{|\alpha|\le m} \frac{f_\alpha(x)}{\alpha!}(z-x)^\alpha,$$

$$(3.2)\qquad (R_x^m F)_\alpha(y) = f_\alpha(y) - \sum_{|\alpha+\beta|\le m} \frac{f_{\alpha+\beta}(x)}{\beta!}(y-x)^\beta$$

A jet F is called a *Whitney jet* on K if it satisfies, for all $m \in N_0$ and $|\alpha| \le m$,

$$(3.3)\qquad |(R_x^m F)_\alpha(y)| = o(|x-y|^{m-|\alpha|})$$

for $x, y \in K$, as $|x-y| \to 0$. We denote by $C^\infty(K)$ for the space of Whitney jets on K

Let $C^m(K), m \in N_0$, be the space of all m times continuously differentiable functions on K in the sense of Whitney i.e.,

$C^m(K) = \{F = (f_\alpha; |\alpha| \leq m) \mid F$ is an array of continuous functions f_α on K such that for each $|\alpha| \leq m$

$$\frac{|(R_x^m F)_\alpha(y)|}{|x-y|^{m-|\alpha|}} \quad \text{tends to zero uniformly as} |x-y| \to 0 \quad \text{in } K\}.$$

Define the norm of $F = (f_\alpha) \in C^m(K)$ by

$$\|F\|_{C^m(K)} = \sup_{|\alpha| \leq m} \|f_\alpha\|_{C(K)}.$$

Then $(C^m(K), \|\cdot\|_{C^m(K)})$ is a Banach space. The Fréchet space $C^\infty(K)$ is defined by

$$C^\infty(K) = \text{proj} \lim_{m \to \infty} C^m(K).$$

Let Ω be an open set in R^n and K a compact subset of Ω. For a weight function w with the properties (α)-(ζ) in Definition 1.1, let φ denote the function defined by (δ). Let φ^* be the Young conjugate of φ.

DEFINITION 3.1. (a) A jet $F = (f_\alpha)$ on K is called a *Whitney jet of Beurling type* if it satisfies the following conditions (3.4) and (3.5): For each $n \in N$, there exist constants A, B such that

$$(3.4) \qquad |f_\alpha(x)| \leq A \exp[n\varphi^*(\frac{|\alpha|}{n})] \text{ for all } \alpha \text{ and } x \in K,$$

$$(3.5) \qquad |(R_x^m F)_\alpha(y)| \leq B \frac{|x-y|^{m-|\alpha|+1}}{(m-|\alpha|+1)!} \exp[n\varphi^*(\frac{m+1}{n})],$$
$$x, y \in K, m \in N_0, |\alpha| \leq m.$$

We write $E_{(\omega)}(K)$ for the space of Whitney jets of Beurling type with the projective limit topology.

(b) A jet $F = (f_\alpha)$ on K is called a *Whitney jet of Roumieu type* if it satisfies the following conditions :

$$(3.6) \qquad |f_\alpha(x)| \leq A \exp[\frac{1}{n}\varphi^*(n|\alpha|)] \text{ for all } \alpha \text{ and } x \in K,$$

$$(3.7) \qquad |(R_x^m F)_\alpha(y)| \leq B \frac{|x-y|^{m-|\alpha|+1}}{(m-|\alpha|+1)!} \exp[\frac{1}{n}\varphi^*(n(m+1))],$$
$$x, y \in K, m \in N_0, |\alpha| \leq m$$

for some $n \in N$ and some constants $A > 0$, $B > 0$.

We write $E_{\{\omega\}}(K)$ for the space of Whitney jets of Roumieu type with the inductive limit topology.

For a jet $F = (f_\alpha)$ on K and $n \in N$, we define

(3.8)
$$||F||_{K,n} = \inf\{A : \text{Constant } A \text{ satisfies (3.4) for all } x \in K \text{ and } \alpha \in N_0^n\}$$
$$+ \inf\{B : \text{Constant } B \text{ satisfies (3.5) for all } x, y \in K, |\alpha| \leq m, m \in N_0\}.$$

THEOREM 3.2. *We define, for $n \in N$,*

$$E_{(\omega)}(K, n) = \{F = (f_\alpha) \text{ a jet on } K : ||F||_{K,n} < \infty\}, \tag{3.9}$$
$$E_{\{\omega\}}(K, n) = \{F = (f_\alpha) \text{ a jet on } K : ||F||_{K,\frac{1}{n}} < \infty\}. \tag{3.10}$$

Then

$$E_{(\omega)}(K) = \text{proj} \lim_{n\to\infty} E_{(\omega)}(K, n), \tag{3.11}$$
$$E_{\{\omega\}}(K) = \text{ind} \lim_{n\to\infty} E_{\{\omega\}}(K, n). \tag{3.12}$$

PROOF. They are obvious

Let $M = (M_k)_{k \in N_0}$ be a sequence of positive numbers satisfying (M.1), (M.2) and (M.3).

DEFINITION 3.3. A jet $F = (f_\alpha)$ on K is called a *Whitney jet of class* (M_k)(resp. $\{M_k\}$) if it satisfies the following conditions :

$$|f_\alpha(x)| \leq A h^{|\alpha|} M_{|\alpha|}, x \in K, \alpha \in N_0^n; \tag{3.13}$$
$$|(R_x^m F)_\alpha(y)| \leq B \frac{|x - y|^{m-|\alpha|+1}}{(m - |\alpha| + 1)!} h^{m+1} M_{m+1}, \tag{3.14}$$
$$x, y \in K, m \in N_0, |\alpha| \leq m$$

for some constants $A > 0$, $B > 0$ and every(resp. some) $h > 0$.

We write $E_{(M_k)}(K)$ (resp. $E_{\{M_k\}}(K)$) for the space of Whitney jets of class (M_k)(resp. $\{M_k\}$) with the projective(resp. inductive) limit topology.

For a jet $F = (f_\alpha)$, we define, $h > 0$

(3.15) $P_h(F) = \inf\{A : \text{Constant } A \text{ satisfies (3.13) for all } x \in K, \alpha \in N_0^n\}$
$+ \inf\{B : \text{Constant } B \text{ satisfies (3.14) for } x, y \in K, |\alpha| \le m, m \in N_0\}.$

THEOREM 3.4. *For $n \in N$, we define*

$$E_{(M_k)}(K, n) = \{F = (f_\alpha) \text{ a jet on } K : P_{\frac{1}{n}}(F) < \infty\}, \tag{3.16}$$

$$E_{\{M_k\}}(K, n) = \{F = (f_\alpha) \text{ a jet on } K : P_n(F) < \infty\}. \tag{3.17}$$

Then

$$E_{(M_k)}(K) = \text{proj} \lim_{n\to\infty} E_{(M_k)}(K, n), \tag{3.18}$$

$$E_{\{M_k\}}(K) = \text{ind} \lim_{n\to\infty} E_{\{M_k\}}(K, n). \tag{3.19}$$

PROOF. They are obvious.

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