THE SPACES OF ULTRADIFFERENTIABLE
FUNCTIONS OF TWO TYPES AND WHITNEY JETS

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0. Introduction

We investigate some problems on the space $E_M(\Omega)$ (resp. $E_{[\omega]}(\Omega)$) of ultradifferentiable functions of class $M = (M_k)_{k \in \mathbb{N}_0(=\mathbb{N}\cup\{0\})}$ (resp. Beurling, Roumieu type) and that of $E_M(K)$ (resp. $E_{[\omega]}(K)$) of Whitney jets of class $M$ (resp. Beurling, Roumieu type) on a compact set $K$ in $\mathbb{R}^n$. Also we consider the problems on non quasi-analytic classes of functions. Here $M$ (resp. $[\omega]$) stands for $(M_k)$ or ${M_k}$ (resp. $(\omega)$ or $\{\omega\}$).

1. The spaces $E_{[\omega]}(\Omega)$ and $E_M(\Omega)$

Throughout $N_0$ and $N$ denote the sets of all nonnegative integers and positive integers, respectively. Let $M = (M_k)_{k \in N_0}$ be a sequence of positive numbers which satisfies some of the following conditions with $M_0 = 1$,

(M.1) $M_k^2 \leq M_{k-1}M_{k+1}$, $k \in N$;

(M.2) There are constants $K > 0$ and $H > 1$ such that

$$M_k \leq KH^k \min_{0 \leq l \leq k} M_l M_{k-l},$$

$k \in N_0$;
(M.3) There is a constant $L > 0$ such that

$$\sum_{i=k+1}^{\infty} \frac{M_{l-1}}{M_l} \leq Lk \frac{M_k}{M_{k+1}}, \quad k \in \mathbb{N};$$

(M.3)'

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$$

We write $m_k = \frac{M_k}{M_{k-1}}, k \in \mathbb{N}$, and define

$$m(t) = \text{the number of } m_k \leq t, M(t) = \sup \log \frac{t^k}{M_k}.$$

**Definition 1.1.** Let $w : [0, \infty) \to [0, \infty)$ be a continuous increasing function with $w(0) = 0$ and $\lim_{t \to \infty} w(t) = \infty$. We consider the following conditions on $w$:

(a) $0 = w(0) \leq w(s + t) \leq w(s) + w(t)$ for all $s, t \in [0, \infty)$,

(b) $\int_0^\infty \frac{w(t)}{1 + t^2} dt < \infty$;

(c) $\lim_{t \to \infty} \frac{\log t}{w(t)} = 0$;

(d) $\varphi : t \to w(e^t)$ is convex;

(e) There exists $C > 0$ with $\int_1^\infty \frac{w(yt)}{t^2} dt \leq Cw(y) + C$ for all $y \geq 0$;

(f) There exists $H \geq 1$ with $2w(t) \leq w(Ht) + H$ for all $t \geq 0$.

**Definition 1.2.** A continuous increasing function $w : [0, \infty) \to [0, \infty)$ with $w(0) = 0$ will be called a weight function if it has the properties (γ), (δ) and (ε) in Definition 1.1.

We shall denote by $\varphi^* : [0, \infty) \to [0, \infty)$ the Young conjugate of the convex function $\varphi$ defined in (δ), i.e.,

$$\varphi^*(s) = \sup \{st - \varphi(t) | t \geq 0\}.$$

**Definition 1.3.** Let $\Omega$ be an open set in $\mathbb{R}^n$.

(a) We define the space $E_{(\omega)}(\Omega)$ (resp. $E_{1(\omega)}(\Omega)$) of ultradifferentiable functions of Beurling type (resp. Roumieu type) is the set of $C^\infty$-functions $f$ in $\Omega$ with the property that for each compact set $K \subset \Omega$
and each \( n \in N \) (resp. some \( n \in N \)), \( \|f\|_{K,n} \) (resp. \( \|f\|_{K,\frac{1}{n}} \)) is finite, where
\[
\|f\|_{K,n} = \sup_{x \in K} \frac{|\partial^\alpha f(x)|}{\exp[\varphi^*(\frac{\|x\|}{n})]}.
\]
The topology in the space \( E(\omega)(\Omega) \) (resp. \( E(\omega)(\Omega) \)) is given the projective limit topology over \( K \) and \( n \) (resp. the projective limit topology over \( K \) of the inductive limit over \( n \)).

(b) We define the space \( E(M_k)(\Omega) \) (resp. \( E(M_k)(\Omega) \)) of ultradifferentiable functions of class \( (M_k) \) (resp. \( \{M_k\} \)), i.e., \( E(M_k)(\Omega) \) (resp. \( E(M_k)(\Omega) \)) = \( \{ f \in C^\infty(\Omega) : \text{for each compact set } K \subset \Omega \text{ and each } h > 0, P_{K,h}(f) < \infty \} \), where
\[
P_{K,h}(f) = \sup_{x \in K} \frac{|D^\alpha f(x)|}{h^{\|M|\alpha|}}.
\]
The topology on the space \( E(M_k)(\Omega) \) (resp. \( E(M_k)(\Omega) \)) is given the projective limit topology over \( K \) and \( n \) (resp. the projective limit topology over \( K \) of the inductive limit over \( n \in N \)).

**Theorem 1.4.** For a compact set \( K \subset \Omega \) and \( n \in N \), we define
\[
(1.1) \quad E(\omega)(K,n) = \{ f \in C^\infty(\Omega) : \|f\|_{K,n} < \infty \},
\]
\[
E(\omega)(K,n) = \{ f \in C^\infty(\Omega) : \|f\|_{K,\frac{1}{n}} < \infty \},
\]
\[
E(M_k)(K,n) = \{ f \in C^\infty(\Omega) : P_{K,\frac{1}{n}} < \infty \} \text{ and }
\]
\[
E(M_k)(K,n) = \{ f \in C^\infty(\Omega) : P_{K,n} < \infty \}.
\]
Theorem 1.4. For a compact set \( K \subset \Omega \) and \( n \in N \), we define
\[
(1.2) \quad E(\omega)(\Omega) = \text{proj lim }_{K \in \Omega} \lim_{n \to \infty} E(\omega)(K,n),
\]
\[
E(\omega)(\Omega) = \text{proj lim }_{K \in \Omega} [\text{ind lim }_{n \to \infty} E(\omega)(K,n)],
\]
\[
E(M_k)(\Omega) = \text{proj lim }_{K \in \Omega} E(M_k)(K,n), \text{ and }
\]
\[
E(M_k)(\Omega) = \text{proj lim }_{K \in \Omega} [\text{ind lim }_{n \to \infty} E(M_k)(K,n)].
\]
They are well defined and can be proved by the following remark:

(i) \[
\exp[n\varphi^*(\frac{\alpha}{n})] = \sup_{t \geq 0} \frac{t^{|\alpha|}}{\exp nw(t)},
\]
\[
\exp[\varphi^*(\frac{1}{n}|\alpha|)] = \sup_{t \geq 0} \frac{t^{|\alpha|}}{\exp \frac{1}{n} w(t)}.
\]

(ii) \[
\exp[nw(t)] = \sup_{x \geq 0} \frac{t^x}{\exp[n\varphi^*(\frac{x}{n})]},
\]
\[
\exp[\frac{1}{n} w(t)] = \sup_{x \geq 0} \frac{t^x}{\exp[\frac{1}{n}\varphi^*(nx)]}.
\]

**Theorem 1.5.** The space \(E_{\{M_k\}}(\mathbb{R}^n)\) of ultradifferentiable functions of class \(\{M_k\}\) is a Silva space, that is, inductive limit of Fréchet spaces such that the canonical mappings are compact.

**Proof.** We define, for \(j \in \mathbb{N}\), \(E_{M,j}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{for every compact set } K \text{ in } \mathbb{R}^n, P_{K,j}(f) < \infty\}\), where the topology in \(E_{M,j}(\mathbb{R}^n)\) is defined by, for an increasing sequence \(\{K_i\}\) of compact sets such that \(\bigcup K_i = \mathbb{R}^n\), the system of seminorms \(\{P_{K_i,j} : i \in \mathbb{N}\}\).

Then \(E_{M,j}(\mathbb{R}^n)\) is a Fréchet space with \(\{P_{K_i,j} : i \in \mathbb{N}\}\). We have the well-known properties such that the image of a bounded set under a continuous linear mapping is a bounded set and the closure of a bounded set is bounded. Also the space \(E_{M,j}(\mathbb{R}^n)\) has the property that all bounded and closed subsets are compact by Ascoli’s theorem. Therefore, for \(j < k\), the inclusion mappings \(E_{M,j}(\mathbb{R}^n) \hookrightarrow E_{M,k}(\mathbb{R}^n)\) are compact. Hence we have

\[
E_{\{M_k\}}(\mathbb{R}^n) = \operatorname{ind} \lim_{j \to \infty} E_{M,j}(\mathbb{R}^n).
\]

**Theorem 1.6.** The space \(E_{\{\omega\}}(\mathbb{R}^n)\) of ultradifferentiable functions of Roumieu type is a Silva space.

Indeed, if a sequence \(\{M_k\}_{k \in \mathbb{N}_0}\) of positive numbers satisfies \((M.1)\), \((M.2)\) and \((M.3)\), then there exists a weight function \(w(t)\) satisfying all conditions \((\alpha)-(\zeta)\) in Definition 1.1 such that \(E_{\{M_k\}}(\mathbb{R}^n) = E_{\{\omega\}}(\mathbb{R}^n)\) and vice versa (see [6], Theorems 3.1, 3.2).
2. Non quasi-analyticity

(M.1) is equivalent to saying that the sequence

\[ m_k = \frac{M_k}{M_{k-1}}, \quad k \in \mathbb{N} \]

is increasing. We denote by \( m(t) \) the number of \( m_k \leq t \). Then we have

\[ M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda. \]

**PROPOSITION 2.1.** (1) \( \int_0^\infty \frac{M(t)}{t^2} dt < \infty \Rightarrow \lim_{t \to \infty} \frac{M(t)}{t} = 0. \)

(2) \( \int_0^\infty \frac{m(t)}{t^2} dt < \infty \Rightarrow \lim_{t \to \infty} \frac{m(t)}{t} = 0. \)

**PROOF** (1) For otherwise, there exists a constant \( \epsilon > 0 \) and a sequence \( t_1 < t_2 < \cdots \) such that \( \frac{M(t_j)}{t_j} > \epsilon \) and \( t_{j+1} > 2t_j \). Hence

\[ \int_0^\infty \frac{M(t)}{t^2} dt \geq \sum_{j=1}^{\infty} \int_{t_j}^{2t_j} \frac{M(t_j)}{t^2} dt = \sum_{j=1}^{\infty} \frac{M(t_j)}{2t_j} = \infty. \]

Hence \( \lim_{t \to \infty} \frac{M(t)}{t} = 0 \).

(2) Similarly we can prove it.

**THEOREM 2.2.** \( \lim_{k \to \infty} \frac{k}{m_k} = 0 \Leftrightarrow \lim_{t \to \infty} \frac{m(t)}{t} = 0. \)

**PROOF** (\( \Rightarrow \)) For every \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \frac{k}{m_k} < \epsilon \) when \( k \geq n_0 \), i.e., \( k \epsilon < m_k \) for \( k \geq n_0 \). If \( t = \frac{1}{\epsilon} n_0 \), then \( \frac{m(t)}{t} = \frac{m(\frac{1}{\epsilon} n_0)}{\frac{1}{\epsilon} n_0} \leq \frac{n_0}{\frac{1}{\epsilon} n_0} = \epsilon. \)

(\( \Leftarrow \)) For every \( \epsilon = \frac{1}{n} \), there exists \( M > 0 \) such that \( t \geq M \) implies \( \frac{m(t)}{t} < \frac{1}{n} \), i.e., there exists \( n_0 \in \mathbb{N} \) such that \( t \geq n_0 \) implies \( \frac{m(n_0 + k)}{n_0 + k} < \frac{1}{n} \). If \( k = n_0 \), then \( \frac{k}{m_k} = \frac{n_0}{m_n} \leq \frac{n_0}{n_0} = \frac{1}{n} \to 0 \) as \( n \to \infty. \)

**PROPOSITION 2.3.** \( \lim_{t \to \infty} \frac{m(t)}{t} = 0 \Leftrightarrow \lim_{t \to \infty} \frac{M(t)}{t} = 0. \)
PROOF. Obvious by L'Hôpital's law.

**Proposition 2.4.** ([7], Proposition 3.1) Suppose that \( M = (M_k)_{k \in \mathbb{N}_0} \) satisfies (M.1) and (M.3)', then

\[
(1) \quad \int_0^\infty \frac{M(t)}{t^2} \, dt = \int_0^\infty \frac{m(t)}{t^2} \, dt,
\]
\[
(2) \quad \int_0^\infty \frac{dm(t)}{t} = \int_0^\infty \frac{m(t)}{t^2} \, dt.
\]

**Proof.** See [7].

**Proposition 2.5.** ([7], Proposition 3.5) Suppose that \( M = (M_k)_{k \in \mathbb{N}_0} \) satisfies (M.1) and (M.3)'. Then we have the following relations:

\[
(1) \quad \int_0^t \frac{m(\lambda)}{\lambda^2} \, d\lambda = \frac{M(t)}{t} + \int_0^t \frac{M(\lambda)}{\lambda^2} \, d\lambda,
\]
\[
(2) \quad \int_t^\infty \frac{M(\lambda)}{\lambda^2} \, d\lambda = \frac{M(t)}{t} + \int_t^\infty \frac{m(\lambda)}{\lambda^2} \, d\lambda, \text{ and hence by (1) or (2)}
\]
\[
(3) \quad \int_0^\infty \frac{M(\lambda)}{\lambda^2} \, d\lambda = \int_0^\infty \frac{m(\lambda)}{\lambda^2} \, d\lambda. \text{ By (1) and (2) we have}
\]
\[
(4) \quad \int_0^t \frac{m(\lambda) - M(\lambda)}{\lambda^2} \, d\lambda = \int_t^\infty \frac{M(\lambda) - m(\lambda)}{\lambda^2} \, d\lambda.
\]

**Proof.** See [7].

Suppose that \( M = (M_k)_{k \in \mathbb{N}_0} \) satisfies (M.1). Then \( M \) satisfies (M.3) if and only if there is a constant \( A \) such that

\[
(*) \quad \int_t^\infty \frac{m(\lambda)}{\lambda^2} \, d\lambda \leq (A + 1) \frac{m(t)}{t}
\]

for \( t \geq m_1 \) (see Komatsu [4], Proposition 4.4).

Integrating both sides of (*), we obtain for \( t \geq m_1 \)

\[
\int_{m_1}^t d\mu \int_\mu^\infty \frac{m(\lambda)}{\lambda^2} \, d\lambda = t \int_0^\infty \frac{m(\lambda)}{\lambda^2} \, d\lambda + \int_t^\infty \frac{m(\lambda)}{\lambda} \, d\lambda - m_1 \int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2} \, d\lambda
\]
\[
\leq (A + 1) \int_{m_1}^t \frac{m(\mu)}{\mu} \, d\mu = (A + 1) M(t).
\]
Hence we have the following relation:

\[(1) \quad t \int_{t}^{\infty} \frac{m(\lambda)}{\lambda^2} d\lambda \leq A \int_{m_1}^{t} \frac{m(\lambda)}{\lambda} d\lambda + m_1 \int_{m_1}^{\infty} \frac{m(\lambda)}{\lambda^2} d\lambda = AM(t) + m_1 \int_{0}^{\infty} \frac{m(\lambda)}{\lambda^2} d\lambda.\]

By Proposition 2.5(2), we have \(t \int_{t}^{\infty} \frac{M(\lambda)}{\lambda^2} d\lambda = M(t) + t \int_{t}^{\infty} \frac{M(\lambda)}{\lambda^2} d\lambda.\)

Hence, by (1) and Proposition 2.5(3), we have the relation:

\[(2) \quad t \int_{t}^{\infty} \frac{M(\lambda)}{\lambda^2} d\lambda \leq (A + 1)M(t) + m_1 \int_{0}^{\infty} \frac{M(\lambda)}{\lambda^2} d\lambda.\]

### 3. Whitney jets of class \(M\) on \(K\)

The letters \(\alpha, \beta\) will mean multi-indexes in \(N_0^n\). For \(\alpha = (\alpha_1, \cdots, \alpha_n)\), we write \(\alpha! = \alpha_1! \cdots \alpha_n!\) and \(|\alpha| = \alpha_1 + \cdots + \alpha_n\). Also, \(\alpha \leq \beta\) stands for \(\alpha_i \leq \beta_i (i = 1, \cdots, n)\) and, for \(x \in R^n, x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\). Let \(K\) be a compact set in \(R^n\).

A jet in \(K\) is a multisequence \(F = (f_\alpha)\) of continuous functions \(f_\alpha\) on \(K\). For a jet \(F\), for \(x, y \in K, z \in R^n, m \in N_0\) and \(|\alpha| \leq m\), we put

\[(3.1) \quad (T^m_x F)(z) = \sum_{|\alpha| \leq m} \frac{f_\alpha(x)}{\alpha!} (z - x)^\alpha,\]

\[(3.2) \quad (R^m_x F)_{\alpha}(y) = f_\alpha(y) - \sum_{|\alpha + \beta| \leq m} \frac{f_{\alpha + \beta}(x)}{\beta!} (y - x)^\beta\]

A jet \(F\) is called a Whitney jet on \(K\) if it satisfies, for all \(m \in N_0\) and \(|\alpha| \leq m\),

\[(3.3) \quad |(R^m_x F)_{\alpha}(y)| = o(|x - y|^{m - |\alpha|})\]

for \(x, y \in K\), as \(|x - y| \to 0\). We denote by \(C^\infty(K)\) for the space of Whitney jets on \(K\).
Let $C^m(K), m \in \mathbb{N}_0$, be the space of all $m$ times continuously differentiable functions on $K$ in the sense of Whitney i.e.,

$$C^m(K) = \{ F = (f_\alpha; |\alpha| \leq m) \mid F \text{ is an array of continuous functions } f_\alpha \text{ on } K \text{ such that for each } |\alpha| \leq m $$

$$\frac{|(R^m_x F)_\alpha(y)|}{|x - y|^{m-|\alpha|}} \text{ tends to zero uniformly as } |x - y| \to 0 \text{ in } K \}.$$ 

Define the norm of $F = (f_\alpha) \in C^m(K)$ by

$$\| F \|_{C^m(K)} = \sup_{|\alpha| \leq m} \| f_\alpha \|_{C(K)}.$$

Then $(C^m(K), \| \cdot \|_{C^m(K)})$ is a Banach space. The Fréchet space $C^\infty(K)$ is defined by

$$C^\infty(K) = \lim_{m \to \infty} C^m(K).$$

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $K$ a compact subset of $\Omega$. For a weight function $w$ with the properties (a)-(c) in Definition 1.1, let $\varphi$ denote the function defined by (d). Let $\varphi^*$ be the Young conjugate of $\varphi$.

DEFINITION 3.1. (a) A jet $F = (f_\alpha)$ on $K$ is called a Whitney jet of Beurling type if it satisfies the following conditions (3.4) and (3.5):

For each $n \in \mathbb{N}$, there exist constants $A, B$ such that

(3.4) $|f_\alpha(x)| \leq A \exp[n\varphi^*\left(\frac{|\alpha|}{n}\right)]$ for all $\alpha$ and $x \in K$,

(3.5) $|(R^m_x F)_\alpha(y)| \leq B \frac{|x - y|^{m-|\alpha|+1}}{(m - |\alpha| + 1)!} \exp[n\varphi^*(\frac{m + 1}{n})]$, $x, y \in K, m \in \mathbb{N}_0, |\alpha| \leq m$.

We write $E_\omega(K)$ for the space of Whitney jets of Beurling type with the projective limit topology.

(b) A jet $F = (f_\alpha)$ on $K$ is called a Whitney jet of Roumieu type if it satisfies the following conditions:

(3.6) $|f_\alpha(x)| \leq A \exp[\frac{1}{n}\varphi^*(n|\alpha|)]$ for all $\alpha$ and $x \in K$,

(3.7) $|(R^m_x F)_\alpha(y)| \leq B \frac{|x - y|^{m-|\alpha|+1}}{(m - |\alpha| + 1)!} \exp[\frac{1}{n}\varphi^*(n(m + 1))]$, $x, y \in K, m \in \mathbb{N}_0, |\alpha| \leq m$. 

for some \( n \in \mathbb{N} \) and some constants \( A > 0, B > 0 \).

We write \( E_{(\omega)}(K) \) for the space of Whitney jets of Roumieu type with the inductive limit topology.

For a jet \( F = (f_\alpha) \) on \( K \) and \( n \in \mathbb{N} \), we define

\[
\| F \|_{K, n} = \inf \{ A : \text{Constant } A \text{ satisfies (3.4) for all } x \in K \text{ and } \alpha \in \mathbb{N}_0^n \} + \inf \{ B : \text{Constant } B \text{ satisfies (3.5) for all } x, y \in K, |\alpha| \leq m, m \in \mathbb{N}_0 \}.
\]

**Theorem 3.2.** We define, for \( n \in \mathbb{N} \),

\[
\begin{align*}
E_{(\omega)}(K, n) &= \{ F = (f_\alpha) \text{ a jet on } K : \| F \|_{K, n} < \infty \}, \\
E_{(\omega)}(K) &= \{ F = (f_\alpha) \text{ a jet on } K : \| F \|_{K, \frac{1}{n}} < \infty \}.
\end{align*}
\]

Then

\[
\begin{align*}
E_{(\omega)}(K) &= \text{proj } \lim_{n \to \infty} E_{(\omega)}(K, n), \\
E_{(\omega)}(K) &= \text{ind } \lim_{n \to \infty} E_{(\omega)}(K, n).
\end{align*}
\]

**Proof.** They are obvious

Let \( M = (M_k)_{k \in \mathbb{N}_0} \) be a sequence of positive numbers satisfying (M.1), (M.2) and (M.3).

**Definition 3.3.** A jet \( F = (f_\alpha) \) on \( K \) is called a *Whitney jet of class* \((M_k)\) (resp. \( \{ M_k \} \)) if it satisfies the following conditions:

\[
\begin{align*}
|f_\alpha(x)| &\leq Ah^{|\alpha|}M_{|\alpha|}, x \in K, \alpha \in \mathbb{N}_0^n; \\
|(D_{x}^{m}F)_{\alpha}(y)| &\leq B \frac{|x - y|^{n-|\alpha|+1}}{(m-|\alpha|+1)!} h^{m+1}M_{m+1}, \\
x, y \in K, m \in \mathbb{N}_0, |\alpha| \leq m
\end{align*}
\]

for some constants \( A > 0, B > 0 \) and every (resp. some) \( h > 0 \).
We write $E_{(M_k)}(K)$ (resp. $E_{(M_k)}(K)$) for the space of Whitney jets of class $(M_k)$ (resp. $(M_k)$) with the projective (resp. inductive) limit topology.

For a jet $F = (f_\alpha)$, we define, $h > 0$

$$P_h(F) = \inf\{A : \text{Constant } A \text{ satisfies } (3.13) \text{ for all } x \in K, \alpha \in N_0^n\}$$

$$+ \inf\{B : \text{Constant } B \text{ satisfies } (3.14) \text{ for } x, y \in K, |\alpha| \leq m, m \in N_0\}.$$

**Theorem 3.4.** For $n \in N$, we define

$$(3.16) \quad E_{(M_k)}(K, n) = \{F = (f_\alpha) \text{ a jet on } K : P_{\frac{1}{n}}(F) < \infty\},$$

$$(3.17) \quad E_{(M_k)}(K, n) = \{F = (f_\alpha) \text{ a jet on } K : P_{n}(F) < \infty\}.$$ 

Then

$$(3.18) \quad E_{(M_k)}(K) = \lim_{n \to \infty} E_{(M_k)}(K, n),$$

$$(3.19) \quad E_{(M_k)}(K) = \lim_{n \to \infty} E_{(M_k)}(K, n).$$

**Proof.** They are obvious.

**References**


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