SOME GENERALIZED FIXED POINT THEOREMS

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1. Introduction

In 1974, Subrahmanyam[4] proved the following fixed point theorem of contraction principle. Let \((X,d)\) be a complete metric space and let \(T\) be a continuous mapping from \(X\) into itself. Suppose that there exists \(r \in [0,1)\) such that
\[
d(Tx, T^2x) \leq r \cdot d(x, Tx) \quad \text{for} \quad x \in X.
\]
Then there exists \(z \in X\) such that \(z = Tz\).

In 1996, Kada, Suzuki and Takahashi[2] first introduced the concept of \(w\)-distance on a metric space and generalized Subrahmanyam fixed point theorem by weakening the metric \(d\) and the continuity of \(T\) with a \(w\)-distance \(p\) and some condition, respectively, as follows: Let \(X\) be a complete metric space, \(p\) a \(w\)-distance on \(X\) and \(T\) a mapping from \(X\) into itself. Suppose that there exists \(r \in [0,1)\) such that
\[
p(Tx, T^2x) \leq r \cdot p(x, Tx) \quad \text{for} \quad x \in X \quad (\ast)
\]
and that
\[
\inf \left\{ p(x, y) + p(x, Tx) : x \in X \right\} > 0 \quad (\ast\ast)
\]
for every \(y \in X\) with \(y \neq Ty\). Then there exists \(z \in X\) such that \(z = Tz\). This theorem also generalizes the fixed point theorems of Kannan[3] and Ćirić[1].

Ume[5] improved fixed point theorems of Kannan[3], Ćirić[1] and Kada, Suzuki, and Takashi[2] by using the condition \((\ast\ast)\) for more

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general contractive mapping than quasi-contractive mapping satisfying the condition (*).

In this paper, we obtain some generalizations of Subrahmanyam fixed point theorem and Ćirić fixed point theorem using the concept of $w$-distance.

2. Preliminaries

**Definition 2.1.** Let $X$ be a metric space with a metric $d$. Then a function $p : X \times X \rightarrow [0, \infty)$ is called a $w$-distance on $X$ if the following are satisfied:

1. $p(x, z) \leq p(x, y) + p(y, z)$ for any $x$, $y$, $z \in X$;
2. for any $x \in X$, $p(x, \cdot) : X \rightarrow [0, \infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Consider some examples of a $w$-distance.

**Example 1** [2]. Let $X$ be a metric space with a metric $d$. Then $p = d$ is a symmetric (i.e., $p(x, y) = p(y, x)$ for $x, y \in X$) $w$-distance on $X$.

Above example shows the possibility that a $w$-distance $p$ is to be a very useful tool to generalize several fixed point theorems.

**Example 2.** Let $X$ be a metric space and $p$ a $w$-distance on $X$. Then the function $q : X \times X \rightarrow [0, \infty)$ defined by

$$q(x, y) = \max\{p(x, y), p(y, x)\}$$

for $x, y \in X$ is a symmetric $w$-distance on $X$, provided that $p(\cdot, x) : X \rightarrow [0, \infty]$ is lower semicontinuous.

**Proof.** Let $x$, $y$, $z \in X$. Then we have

$$p(x, y) \leq p(x, z) + p(z, y) \leq \max\{p(x, z), p(z, x)\} + \max\{p(z, y), p(y, z)\} = q(x, z) + q(z, y)$$

Similarly,

$$p(y, x) \leq p(y, z) + p(z, x) \leq q(z, y) + q(x, z).$$
Thus
\[ q(x, y) = \max\{p(x, y), p(y, x)\} \leq q(x, z) + q(z, y), \]
which satisfies (1). For each \( c \in \mathbb{R} \) and \( x \in X \),
\[ \{ y \mid q(x, y) > c \} = \{ y \mid p(x, y) > c \} \cup \{ y \mid p(y, x) > c \}. \]

Since \( p(x, \cdot) \) and \( p(\cdot, x) \) are lower semicontinuous, \( \{ y \in X \mid q(x, y) > c \} \) is open in \( X \), which shows that \( q(x, \cdot) \) is a lower semicontinuous function satisfying (2). By using the condition (3) for \( p \), we can easily show the condition (3) for \( q \).

Remark that a \( w \)-distance \( p : X \times X \to [0, \infty) \) defined by \( p(x, y) = \|y\| \) in [2] is not symmetric.

**Example 3.** Let \( X \) be a normed linear space. Then a function \( p : X \times X \to [0, \infty) \) defined by
\[ p(x, y) = \max\{\|x\|, \|y\|\} \]
for \( x, y \in X \), is a symmetric \( w \)-distance on \( X \).

**Lemma 2.2[2]** Let \( X \) be a metric space with a metric \( d \) and \( p \) a \( w \)-distance on \( X \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \), \( \{\alpha_n\} \) and \( \{\beta_n\} \) sequences in \( [0, \infty) \) converging to 0, and \( x, y, z \in X \). Then the following hold:

(i) if \( p(x_n, y) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in N \), then \( y = z \), in particular, if \( p(x, y) = 0 \) and \( p(x, z) = 0 \), then \( y = z \);  
(ii) if \( p(x_n, y_n) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in N \), then \( \{y_n\} \) converges to \( z \);  
(iii) if \( p(x_n, x_m) \leq \alpha_n \) for any \( n, m \in N \) with \( m > n \), then \( \{x_n\} \) is a Cauchy sequence;  
(iv) if \( p(y, x_n) \leq \alpha_n \) for any \( n \in N \), then \( \{x_n\} \) is a Cauchy sequence.

3. A generalization of Subrahmanyam fixed point theorem

In this section, we generalize Subrahmanyam fixed point theorem using the concept of a \( w \)-distance.
**Theorem 3.1.** Let $T$ be a mapping from a metric space $X$ into itself and $p$ a $w$-distance on $X$. Suppose that there exists a point $u \in X$ such that

1. \( \{T^m u\} \) has a convergent subsequence with a limit $z \in X$;
2. $p(Tx, T^2 x) \leq r \cdot p(x, Tx)$ for $x \in O(u)$, where $O(u) = \{u, Tu, T^2 u, \ldots\}$ and $r \in [0, 1)$;
3. a function $G : X \rightarrow \mathbb{R}$ defined by $G(x) = p(x, Tx)$ is lower semicontinuous at $z \in X$;
4. $p(z, x) = p(x, z)$ for each $x \in O(u)$.

Then $z$ is a fixed point of $T$.

**Proof.** Set $x = T u$. Then by (2),

$$p(T^2 u, T^3 u) \leq r \cdot p(T u, T^2 u) \leq r^2 \cdot p(u, T u) \leq \cdots.$$  

By induction, we have

$$p(T^n u, T^{n+1} u) \leq r^n \cdot p(u, T u) \quad \text{for any } \quad n \in \mathbb{N}.$$  

If $m > n$,

$$p(T^m u, T^m u) \leq p(T^m u, T^{m+1} u) + \cdots + p(T^{m-1} u, T^m u)$$

$$\leq r^n \cdot p(u, T u) + \cdots + r^{m-1} \cdot p(u, T u)$$

$$\leq \frac{r^n}{1 - r} \cdot p(u, T u).$$

From Lemma 2.2, it implies that $\{T^n u\}$ is a Cauchy sequence. By (1), $\{T^n u\}$ converges to $z$. Since $G(x) = p(x, Tx)$ is lower semicontinuous at $z$, we have

$$p(z, T z) \leq \liminf_{n \to \infty} p(T^n u, T^{n+1} u)$$

$$\leq \liminf_{n \to \infty} r^n \cdot p(u, T u)$$

$$= 0$$

On the other hand, since $p(T^n u, \cdot)$ is lower semicontinuous at $z$,

$$p(T^n u, z) \leq \liminf_{m \to -\infty} p(T^n u, T^m u)$$

$$\leq \frac{r^n}{1 - r} \cdot p(u, T u)$$
From (4) and the fact that $p(x, \cdot)$ is lower semicontinuous,

$$p(z, z) \leq \liminf_{n \to \infty} p(z, T^n u)$$

$$= \liminf_{n \to \infty} p(T^n u, z)$$

$$\leq \liminf_{n \to \infty} \frac{r^n}{1 - r} \cdot p(u, Tu)$$

$$= 0$$

Thus $p(z, z) = 0$ and $p(z, Tz) = 0$. By Lemma 2.2 (i), we have $z = Tz$.

**Theorem 3.2.** Let $T$ be a mapping from a complete metric space $X$ into itself and $p$ a symmetric $w$-distance on $X$. Suppose that there exists $r \in [0, 1)$ such that

$$p(Tx, T^2 x) \leq r \cdot p(x, Tx)$$

for any $x \in X$ and that $G(x) = p(x, Tx)$ is lower semicontinuous. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $y = Ty$, then $p(y, y) = 0$.

**Proof.** Let $u \in X$. Then $\{T^n u\}$ is a Cauchy sequence. Since $X$ is complete, $\{T^n u\}$ converges to some point $z \in X$. The remainder of the process follows the proof of Theorem 3.1.

As a corollary, we obtain the following result of Subrahmanyam

**Theorem 3.3[4].** Let $X$ be a complete metric space with a metric $d$ and $T$ a continuous mapping from $X$ into itself. Suppose that there exists $r \in [0, 1)$ such that

$$d(Tx, T^2 x) \leq r \cdot d(x, Tx)$$

for $x \in X$. Then there exists $z \in X$ such that $z = Tz$.

**Proof.** Any metric $d$ is a symmetric $w$-distance. And the continuity of $T$ guarantees that $G(x) = d(x, Tz)$ is lower semicontinuous.
4. A generalization of Ćirić fixed point theorem

Ćirić[1] proved a fixed point theorem for a quasi-contractive mapping on a complete metric space. In this section, we generalize Ćirić fixed point theorem using the concept of a \( w \)-distance.

**Lemma 4.1**[5]. Let \( X \) be a metric space with a metric \( d \) and \( p \) a \( w \)-distance on \( X \). Let \( T \) be a mapping of \( X \) into itself satisfying
\[
p(Tx, Ty) \leq r \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}
\]
for all \( x, y \in X \) and some \( r \in [0, 1) \). Then
1. for each \( x \in X \) and \( n \in N \),
\[
p(T^n x, T^{n+1} x) \leq \frac{r^n - 1}{1 - r} \cdot a(x),
\]
where \( a(x) = p(x, x) + p(x, Tx) + p(Tx, x) \),
2. for each \( x \in X \), \( \{T^n x\} \) is a Cauchy sequence.

**Theorem 4.2.** Let \( X \) be a metric space, \( p \) a \( w \)-distance on \( X \), and \( T \) a mapping of \( X \) into itself. Suppose that there exists a point \( u \in X \) such that
1. \( \{T^n u\} \) has a convergent subsequence with a limit \( z \in X \);
2. \( p(Tx, Ty) \leq r \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\} \)
   for all \( x, y \in O(u) \) and some \( r \in [0, 1) \);
3. \( p(z, x) = p(x, z) \) for each element \( x \) of \( O(u) \).
4. \( G(x) = p(x, Tx) \) is lower semicontinuous at \( z \in X \).
Then \( z \) is a fixed point of \( T \) and \( p(z, z) = 0 \).

**Proof.** By Lemma 4.1, \( \{T^n u\} \) is a Cauchy sequence. Since \( \{T^n u\} \) has a convergent subsequence with a limit \( z \), \( \{T^n u\} \) converges to \( z \). If \( n < m \),
\[
p(T^n u, T^m u) \leq p(T^n u, T^{n+1} u) + \cdots + p(T^{m-1} u, T^m u)
\]
\[
\leq \frac{r^{n-1}}{1 - r} \cdot a(u) + \cdots + \frac{r^{m-2}}{1 - r} \cdot a(u)
\]
\[
= \frac{r^{n-1}(1 - r^{m-n})}{(1 - r)^2} \cdot a(u),
\]

where \( a(u) = p(u, u) + p(u, Tu) + p(Tu, u) \). From the fact that \( p(x, \cdot) \) is lower semicontinuous at \( z \), we obtain
\[
p(T^n u, z) \leq \liminf_{m \to \infty} p(T^m u, T^n u)
\]
\[
\leq \liminf_{m \to \infty} \frac{r^{m-1}(1 - r^{m-n})}{(1 - r)^2} \cdot a(u)
\]
\[
= \frac{r^{n-1}}{(1 - r)^2} \cdot a(u),
\]
Again applying the lower semicontinuity of \( p(x, \cdot) \), we obtain
\[
p(z, z) \leq \liminf_{n \to \infty} p(z, T^n u)
\]
\[
= \liminf_{n \to \infty} p(T^n u, z)
\]
\[
\leq \liminf_{n \to \infty} \frac{r^{n-1}}{(1 - r)^2} \cdot a(u)
\]
\[
= 0.
\]
And, the condition (4) induces that \( p(z, Tz) = 0 \) as follows ;
\[
p(z, Tz) \leq \liminf_{n \to \infty} p(T^n u, T^{n+1} u)
\]
\[
\leq \liminf_{n \to \infty} \frac{r^{n-1}}{1 - r} a(u)
\]
\[
= 0.
\]
Therefore, \( p(z, z) = 0 \) and \( p(z, Tz) = 0 \), which implies that \( z = Tz \) by Lemma 2.2 (i).

As a corollary, we obtain the following result of Ćirić[1].

**Theorem 4.3** Let \( X \) be a complete metric space with a metric \( d \) and \( T : X \to X \) a mapping such that for all \( x, y \in X \),
\[
d(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},
\]
for some \( r \in [0,1) \). Then
(1) \( T \) has a unique fixed point \( z \) in \( X \),
(2) \( \lim_n T^n x = z \).
PROOF. Let \( x \in X \). Then, by Lemma 4.1, \( \{T^n x\} \) is a Cauchy sequence. Since \( X \) is complete, \( \{T^n x\} \) converges to some point \( z \in X \), which proves (2). On the other hand, since the metric \( d \) is a symmetric \( w \)-distance, the conditions (1), (2) and (3) in Theorem 4.2 are satisfied. Since \( T^n x \to z \), we have

\[
d(Tz, z) \leq \lim\inf_{n \to \infty} d(Tz, T^n x) \\
\leq \lim\inf_{n \to \infty} r \cdot \max\{d(z, T^{n-1} x), d(z, Tz), d(T^{n-1} x, T^n x), \}
\]

\[
d(z, T^n x), d(T^{n-1} x, Tz)\}
\]

\[
\leq r \cdot d(z, Tz) \\
= r \cdot d(Tz, z)
\]

Thus \( d(Tz, z) = 0 \), which implies that \( z = Tz \). To prove uniqueness, let \( y = Ty \) and \( z = Tz \). Then

\[
d(y, z) = d(Ty, Tz) \\
\leq r \cdot \max\{d(y, z), d(y, Ty), d(z, Tz), d(y, Tz), d(z, Ty)\}
\]

\[
= r \cdot \max\{d(y, z), d(y, y), d(z, z), d(y, z), d(z, y)\}
\]

\[
= r \cdot d(y, z).
\]

Hence \( d(y, z) = 0 \) and \( y = z \).

COROLLARY 4.4. Let \( X \) be a complete metric space, \( p \) a \( w \)-distance on \( X \), and \( T \) a continuous mapping from \( X \) into itself. Suppose that there exists \( r \in [0, 1) \) such that

\[
p(Tx, Ty) \leq r \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}
\]

for \( x, y \in X \). Then \( T \) has a unique fixed point \( z \) and \( p(z, z) = 0 \).

PROOF. Let \( u \in X \), then \( \{T^n u\} \) is a Cauchy sequence in a complete metric space \( X \) by Lemma 4.1. Let \( z \) be the limit of \( \{T^n u\} \). Since \( T \) is continuous, we have

\[
Tz = T(\lim_{n \to \infty} T^n u) = \lim_{n \to \infty} T^{n+1} u = z.
\]
Therefore, $z$ is a fixed point of $T$. Assume that $T(y) = y$ and $T(z) = z$ for some $y, z \in X$. Then

$$p(y, y) = p(Ty, Ty) \leq r \cdot \max p(y, y),$$

which shows that $p(y, y) = 0$. Also,

$$p(y, z) = p(Ty, Tz) \leq r \cdot \max \{p(y, z), p(z, y)\}$$

and

$$p(z, y) = p(Tz, Ty) \leq r \cdot \max \{p(z, y), p(y, z)\}.$$ 

If $p(y, z) \leq p(z, y)$ then $p(z, y) \leq p(z, y)$, which implies $p(z, y) = 0$. Since $p(z, z) = 0$ and $p(z, y) = 0$, by Lemma 2.2 we have $z = y$

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