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SOME GENERALIZED FIXED POINT THEOREMS

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1. Introduction

In 1974, Subrahmanyam[4] proved the following fixed point theorem of contraction principle . Let (X,d) be a complete metric space and let T be a continuous mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that

$$d(Tx,T^2x) \leq r \cdot d(x,Tx) \quad \text{for } x \in X.$$

Then there exists $z \in X$ such that $z = Tz$.

In 1996, Kada, Suzuki and Takahashi[2] first introduced the concept of w-distance on a metric space and generalized Subrahmanyam fixed point theorem by weakening the metric d and the continuity of T with a w-distance p and some condition, respectively, as follows : Let X be a complete metric space, p a w-distance on X and T a mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that

$$p(Tx,T^2x) \leq r \cdot p(x,Tx) \quad \text{for} \quad x \in X \tag{$*$}$$

and that

$$\inf\{p(x,y) + p(x,Tx) : x \in X\} > 0 \tag{$**$}$$

for every $y \in X$ with $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. This theorem also generalizes the fixed point theorems of Kannan[3] and Ćirić[1].

Ume[5] improved fixed point theorems of Kannan[3], Ćirić[1] and Kada, Suzuki, and Takashi[2] by using the condition $(**)$ for more

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general contractive mapping than quasi-contractive mapping satisfying the condition $(*)$.

In this paper, we obtain some generalizations of Subrahmanyam fixed point theorem and Ćirić fixed point theorem using the concept of w-distance.

2. Preliminaries

DEFINITION 2.1. Let X be a metric space with a metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x,\ y,\ z \in X$;

(2) for any $x \in X,\ p(x,\cdot)\ :\ X \to [0,\infty)$ is lower semicontinuous ;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

Consider some examples of a w-distance.

EXAMPLE 1[2]. Let X be a metric space with a metric d. Then $p = d$ is a symmetric (*i.e.*, $p(x,y) = p(y,x)$ for $x, y \in X$) w-distance on X.

Above example shows the possibility that a w-distance p is to be a very useful tool to generalize several fixed point theorems.

EXAMPLE 2. Let X be a metric space and p a w-distance on X Then the function $q : X \times X \to [0, \infty)$ defined by

$$q(x,y) = \max\{p(x,y),\ p(y,x)\} \text{ for } x, y \in X$$

is a symmetric w-distance on X, provided that $p(\cdot, x)\ : X \to [0, \infty]$ is lower semicontinuous.

PROOF Let $x,\ y,\ z \in X$. Then we have

$$\begin{aligned} p(x,y) &\leq p(x,z) + p(z,y) \\ &\leq \max\{p(x,z), p(z,x)\} + \max\{p(z,y), p(y,z)\} \\ &= q(x,z) + q(z,y) \end{aligned}$$

Similarly,

$$p(y,x) \leq p(y,z) + p(z,x) \leq q(z,y) + q(x,z).$$

Thus

$$q(x,y) = \max\{p(x,y), p(y,x)\} \le q(x,z) + q(z,y),$$

which satisfies (1). For each $c \in \mathbb{R}$ and $x \in X$,

$$\{y \mid q(x,y) > c\} = \{y \mid p(x,y) > c\} \cup \{y \mid p(y,x) > c\}.$$

Since $p(x,\cdot)$ and $p(\cdot,x)$ are lower semicontinuous, $\{y \in X \mid q(x,y) > c\}$ is open in X, which shows that $q(x,\cdot)$ is a lower semicontinuous function satisfying (2). By using the condition (3) for p, we can easily show the condition (3) for q.

Remark that a w-distance $p : X \times X \to [0,\infty)$ defined by $p(x,y) = \|y\|$ in [2] is not symmetric.

EXAMPLE 3. Let X be a normed linear space. Then a function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = \max\{\|x\|, \|y\|\} \quad \text{for } x, y \in X,$$

is a symmetric w-distance on X

LEMMA 2.2[2] *Let X be a metric space with a metric d and p a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ sequences in $[0,\infty)$ converging to 0, and $x, y, z \in X$. Then the following hold :*

(i) *if $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in N$, then $y = z$, in particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*

(ii) *if $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z ;*

(iii) *if $p(x_n, x_m) \le \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence ;*

(iv) *if $p(y, x_n) \le \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

3. A generalization of Subrahmanyam fixed point theorem

In this section, we generalize Subrahmanyam fixed point theorem using the concept of a w-distance.

THEOREM 3.1. *Let T be a mapping from a metric space X into itself and p a w-distance on X. Suppose that there exists a point $u \in X$ such that*

(1) $\{T^n u\}$ has a convergent subsequence with a limit $z \in X$;
(2) $p(Tx, T^2x) \leq r \cdot p(x, Tx)$ for $x \in O(u)$,
where $O(u) = \{u,\ Tu,\ T^2u,\ \cdots\}$ and $r \in [0, 1)$;
(3) a function $G : X \to \mathbb{R}$ defined by $G(x) = p(x, Tx)$ is lower semicontinuous at $z \in X$;
(4) $p(z, x) = p(x, z)$ for each $x \in O(u)$.
Then z is a fixed point of T.

PROOF. Set $x = Tu$. Then by (2),

$$p(T^2u, T^3u) \leq r \cdot p(Tu, T^2u) \leq r^2 \cdot p(u, Tu) \leq \cdots .$$

By induction, we have

$$p(T^n u, T^{n+1}u) \leq r^n \cdot p(u, Tu) \quad \text{for any} \quad n \in N.$$

If $m > n$,

$$\begin{aligned} p(T^n u, T^m u) &\leq p(T^n u, T^{n+1}u) + \cdots + p(T^{m-1}u, T^m u) \\ &\leq r^n \cdot p(u, Tu) + \cdots + r^{m-1} \cdot p(u, Tu) \\ &\leq \frac{r^n}{1-r} \cdot p(u, Tu). \end{aligned}$$

From Lemma 2 2, it implies that $\{T^n u\}$ is a Cauchy sequence. By (1), $\{T^n u\}$ converges to z. Since $G(x) = p(x, Tx)$ is lower semicontinuous at z, we have

$$\begin{aligned} p(z, Tz) &\leq \liminf_{n\to\infty}\ p(T^n u, T^{n+1}u) \\ &\leq \liminf_{n\to\infty}\ r^n \cdot p(u, Tu) \\ &= 0 \end{aligned}$$

On the other hand, since $p(T^n u, \cdot)$ is lower semicontinuous at z,

$$\begin{aligned} p(T^n u, z) &\leq \lim_{m\to\infty} \inf\ p(T^n u, T^m u) \\ &\leq \frac{r^n}{1-r} \cdot p(u, Tu) \end{aligned}$$

From (4) and the fact that $p(z, \cdot)$ is lower semicontinuous,

$$\begin{aligned} p(z,z) &\leq \liminf_{n\to\infty} \; p(z, T^n u) \\ &= \liminf_{n\to\infty} \; p(T^n u, z) \\ &\leq \liminf_{n\to\infty} \; \frac{r^n}{1-r} \cdot p(u, Tu) \\ &= 0 \end{aligned}$$

Thus $p(z,z) = 0$ and $p(z, Tz) = 0$. By Lemma 2.2 (i), we have $z = Tz$.

THEOREM 3.2. *Let T be a mapping from a complete metric space X into itself and p a symmetric w-distance on X Suppose that there exists $r \in [0,1)$ such that*

$$p(Tx, T^2x) \leq r \cdot p(x, Tx)$$

for any $x \in X$ and that $G(x) = p(x, Tx)$ is lower semicontinuous. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $y = Ty$, then $p(y,y) = 0$.

PROOF. Let $u \in X$. Then $\{T^n u\}$ is a Cauchy sequence. Since X is complete, $\{T^n u\}$ converges to some point $z \in X$. The remainder of the process follows the proof of Theorem 3 1.

As a corollary, we obtain the following result of Subrahmanyam

THEOREM 3.3[4]. *Let X be a complete metric space with a metric d and T a continuous mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that*

$$d(Tx, T^2x) \leq r \cdot d(x, Tx)$$

for $x \in X$. Then there exists $z \in X$ such that $z = Tz$.

PROOF. Any metric d is a symmetric w-distance. And the continuity of T guarantees that $G(x) = d(x, Tx)$ is lower semicontinuous.

4. A generalization of Ćirić fixed point theorem

Ćirić[1] proved a fixed point theorem for a quasi-contractive mapping on a complete metric space. In this section, we generalize Ćirić fixed point theorem using the concept of a w-distance.

LEMMA 4.1[5]. *Let X be a metric space with a metric d and p a w-distance on X. Let T be a mapping of X into itself satisfying*

$$p(Tx,Ty) \leq r \cdot \max\{p(x,y),\ p(x,Tx),\ p(y,Ty),\ p(x,Ty),\ p(y,Tx)\}$$

for all $x, y \in X$ and some $r \in [0,1)$. Then

(1) for each $x \in X$ and $n \in N$,

$$p(T^n x, T^{n+1}x) \leq \frac{r^{n-1}}{1-r}\, a(x),$$

where $a(x) = p(x,x) + p(x,Tx) + p(Tx,x)$, and

(2) for each $x \in X$, $\{T^n x\}$ is a Cauchy sequence.

THEOREM 4.2. *Let X be a metric space, p a w-distance on X, and T a mapping of X into itself. Suppose that there exists a point $u \in X$ such that*

(1) $\{T^n u\}$ has a convergent subsequence with a limit $z \in X$;

(2) $p(Tx,Ty) \leq r \cdot \max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\}$ for all $x, y \in O(u)$ and some $r \in [0,1)$;

(3) $p(z,x) = p(x,z)$ for each element x of $O(u)$.

(4) $G(x) = p(x,Tx)$ is lower semicontinuous at $z \in X$.

Then z is a fixed point of T and $p(z,z) = 0$.

PROOF. By Lemma 4.1, $\{T^n u\}$ is a Cauchy sequence. Since $\{T^n u\}$ has a convergent subsequence with a limit z, $\{T^n u\}$ converges to z. If $n < m$,

$$\begin{aligned} p(T^n u, T^m u) &\leq p(T^n u, T^{n+1}u) + \cdots + p(T^{m-1}u, T^m u) \\ &\leq \frac{r^{n-1}}{1-r} \cdot a(u) + \cdots + \frac{r^{m-2}}{1-r} \cdot a(u) \\ &= \frac{r^{n-1}(1-r^{m-n})}{(1-r)^2} \cdot a(u), \end{aligned}$$

where $a(u) = p(u,u) + p(u,Tu) + p(Tu,u)$. From the fact that $p(x,\cdot)$ is lower semicontinuous at z, we obtain

$$\begin{aligned} p(T^n u, z) &\le \liminf_{m\to\infty} \ p(T^n u, T^m u) \\ &\le \liminf_{m\to\infty} \ \frac{r^{n-1}(1-r^{m-n})}{(1-r)^2} \cdot a(u) \\ &= \frac{r^{n-1}}{(1-r)^2} \cdot a(u), \end{aligned}$$

Again applying the lower semicontinuity of $p(x,\cdot)$, we obtain

$$\begin{aligned} p(z,z) &\le \liminf_{n\to\infty} \ p(z, T^n u) \\ &= \liminf_{n\to\infty} \ p(T^n u, z) \\ &\le \liminf_{n\to\infty} \ \frac{r^{n-1}}{(1-r)^2} \cdot a(u) \\ &= 0. \end{aligned}$$

And, the condition (4) induces that $p(z,Tz) = 0$ as follows ;

$$\begin{aligned} p(z,Tz) &\le \liminf_{n\to\infty} \ p(T^n u, T^{n+1} u) \\ &\le \liminf_{n\to\infty} \ \frac{r^{n-1}}{1-r} a(u) \\ &= 0. \end{aligned}$$

Therefore, $p(z,z) = 0$ and $p(z,Tz) = 0$, which implies that $z = Tz$ by Lemma 2 2 (i)

As a corollary, we obtain the following result of Ćirić[1].

THEOREM 4.3 *Let X be a complete metric space with a metric d and $T : X \to X$ a mapping such that for all x, $y \in X$,*

$$d(Tx,Ty) \le r \ \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for some $r \in [0,1)$. Then

(1) T has a unique fixed point z in X,

(2) $\lim_n T^n x = z$.

Proof. Let $x \in X$. Then, by Lemma 4.1, $\{T^n x\}$ is a Cauchy sequence. Since X is complete, $\{T^n x\}$ converges to some point $z \in X$, which proves (2). On the other hand, since the metric d is a symmetric w-distance, the conditions (1), (2) and (3) in Theorem 4.2 are satisfied. Since $T^n x \to z$, we have

$$\begin{aligned} d(Tz, z) &\leq \liminf_{n\to\infty} \ d(Tz, T^n x) \\ &\leq \liminf_{n\to\infty} r \cdot \max\{d(z, T^{n-1}x), d(z, Tz), d(T^{n-1}x, T^n x), \\ &\qquad\qquad d(z, T^n x), d(T^{n-1}x, Tz)\} \\ &\leq r \cdot d(z, Tz) \\ &= r \cdot d(Tz, z) \end{aligned}$$

Thus $d(Tz, z) = 0$, which implies that $z = Tz$. To prove uniqueness, let $y = Ty$ and $z = Tz$ Then

$$\begin{aligned} d(y, z) &= d(Ty, Tz) \\ &\leq r \cdot \max\{d(y, z), d(y, Ty), d(z, Tz), d(y, Tz), d(z, Ty)\} \\ &= r \cdot \max\{d(y, z), d(y, y), d(z, z), d(y, z), d(z, y)\} \\ &= r \cdot d(y, z). \end{aligned}$$

Hence $d(y, z) = 0$ and $y = z$.

Corollary 4.4. *Let X be a complete metric space, p a w-distance on X, and T a continuous mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that*

$$p(Tx, Ty) \leq r \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}$$

for $x, y \in X$. Then T has a unique fixed point z and $p(z, z) = 0$

Proof. Let $u \in X$, then $\{T^n u\}$ is a Cauchy sequence in a complete metric space X by Lemma 4.1. Let z be the limit of $\{T^n u\}$. Since T is continuous, we have

$$Tz = T(\lim_{n\to\infty} T^n u) = \lim_{n\to\infty} T^{n+1} u = z.$$

Therefore, z is a fixed point of T. Assume that $T(y) = y$ and $T(z) = z$ for some y, $z \in X$. Then

$$p(y, y) = p(Ty, Ty) \leq r \cdot \max p(y, y),$$

which shows that $p(y, y) = 0$. Also,

$$p(y, z) = p(Ty, Tz) \leq r \cdot \max\{p(y, z), p(z, y)\}$$

and

$$p(z, y) = p(Tz, Ty) \leq r \cdot \max\{p(z, y), p(y, z)\}.$$

If $p(y, z) \leq p(z, y)$ then $p(z, y) \leq p(z, y)$, which implies $p(z, y) = 0$. Since $p(z, z) = 0$ and $p(z, y) = 0$, by Lemma 2 2 we have $z = y$

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