East Asian Math J. 15(1999), No 2, pp 313-324

TYPE I_∞ OF A VON NEUMANN ALGEBRA ALG$\mathcal{L}$

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ABSTRACT What we will be concerned with is, first, the question of the condition about $\mathcal{L}$ that gives Alg$\mathcal{L}$ a von Neumann algebra, that is, the question of the condition about $\mathcal{L}$ that will give Alg$\mathcal{L}$ a self-adjoint algebra Secondly, if Alg$\mathcal{L}$ is a von Neumann algebra, we want to find out what type it is.

1. Introduction

The study of self-adjoint operator algebras on Hilbert space is well established, with a long history including some of the strongest mathematicians of the twentieth century. By contrast, non-self-adjoint algebras, particularly reflexive algebras, are only beginning to be studied ; the seminar paper of W. B. Arveson ([1]) in 1974 represents the beginning of widespread interest in reflexive algebras. More recently, such algebras have been found to be of use in physics, in electrical engineering, and in general system theory.

Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. One of the most important classes of such algebras is the sequence Alg$\mathcal{L}_2$, Alg$\mathcal{L}_4$, ..., Alg$\mathcal{L}_\infty$ of "tridiagonal" algebras, discovered by Gilfeather and Larson ([2,4,5]).

We will introduce the terminology which is used in this paper. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{V}$ be a subset of $\mathcal{B}(\mathcal{H})$. $\mathcal{V}$ is called self-adjoint if A^* is in $\mathcal{V}$ for every A in $\mathcal{V}$ (A^* is the adjoint of A). If $\mathcal{V}$ is a vector space over a field

Received November 4, 1998. Revised May 10, 1999
1991 Mathematics Subject Classification 47C15, 47D15

K and if $\mathcal{V}$ is closed under the composition of maps, then $\mathcal{V}$ is called an algebra. $\mathcal{V}$ is called a self-adjoint algebra provided A^* is in $\mathcal{V}$ for every A in $\mathcal{V}$.Otherwise $\mathcal{V}$ is called a non-self-adjoint algebra. $\mathcal{V}$ is a C^*-algebra if $\mathcal{V}$ is a self-adjoint algebra which contains the identity operator I and is closed in the norm topology. $\mathcal{V}$ is a von Neumann algebra if $\mathcal{V}$ is a C^*-algebra which is closed in the weak operator topology.

For any subset $\mathcal{V}$ of $\mathcal{B}(\mathcal{H})$, we shall denote by $\mathcal{V}'$ the commutant of $\mathcal{V}$:

$$\mathcal{V}' = \{B \in \mathcal{B}(\mathcal{H}) : BA = AB \text{ for any } A \in \mathcal{V}\}.$$

For any subset $\mathcal{V}$ of $\mathcal{B}(\mathcal{H})$, $\mathcal{V}'$ is an algebra which contains I in $\mathcal{B}(\mathcal{H})$. If $\mathcal{V}$ is self-adjoint, then $\mathcal{V}'$ is a von Neumann algebra. In particular, if $\mathcal{V}$ is a von Neumann algebra, then $\mathcal{V}'$ is a von Neumann algebra ([8]). Let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\mathcal{V}' \subset \mathcal{B}(\mathcal{H})$ its commutant. Then $\mathcal{V} \cap \mathcal{V}'$ is the common center of the algebras $\mathcal{V}$ and $\mathcal{V}'$. It is obvious that $\mathcal{V} \cap \mathcal{V}' \subset \mathcal{B}(\mathcal{H})$ is a (commutative) von Neumann algebra.

Let $\mathcal{H}$ be a complex Hilbert space. A linear manifold in $\mathcal{H}$ is a subset of $\mathcal{H}$ which is closed under vector addition and under multiplication by complex numbers. A subspace of $\mathcal{H}$ is a linear manifold which is closed in the norm topology. We shall often disregard the distinction between an orthogonal projection and its range space. Thus we consider a subspace lattice as consisting of orthogonal projections or subspaces and we may use the same notation to indicate either.

Given any family $\{X_\alpha\}$ of subspaces of a Hilbert space $\mathcal{H}$, there is a greatest subspace $\wedge X_\alpha$ that is contained in each X_α and a smallest subspace $\vee X_\alpha$ that contains each X_α. Specially, $\wedge X_\alpha = \cap X_\alpha$, while $\vee X_\alpha$ is the subspace $[\cup X_\alpha]$ generated by $\cup X_\alpha$ [7,11].

Let $\mathcal{L}$ be a subset of all orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then $\mathcal{L}$ is called a lattice if $\mathcal{L}$ is closed under the operators "$\wedge$" and "$\vee$", where $E \wedge F$ is the orthogonal projection whose range is $(\text{range}E) \cap (\text{range}F)$ and $E \vee F$ is the orthogonal projection whose range is $[(\text{range}E) \cup (\text{range}F)]$. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, Alg$\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$, that is,

$$\text{Alg}\mathcal{L} = \{A \in \mathcal{B}(\mathcal{H}) : AE = EAE \text{ for any } E \in \mathcal{L}\}.$$

A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on a Hilbert space $\mathcal{H}$, containing the zero operator $\mathbf{0}$ and I. Dually, if $\mathcal{V}$ is a subalgebra of the set of all bounded operators acting on $\mathcal{H}$, then Lat$\mathcal{V}$ is the lattice of all orthogonal projections which leave invariant for each operator in $\mathcal{V}$.

An algebra $\mathcal{V}$ is reflexive if $\mathcal{V} = \text{AlgLat}\mathcal{V}$. A lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = \text{LatAlg}\mathcal{L}$. A lattice $\mathcal{L}$ is commutative if each pair of orthogonal projections in $\mathcal{L}$ commutes. Subspace lattices need not be reflexive , however, commutative ones are reflexive ([1])

Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then

(a) Alg$\mathcal{L}$ is an algebra containing I ;
(b) Alg$\mathcal{L}$ is closed in the norm topology ;
(c) Alg$\mathcal{L}$ is closed in the weak operator topology

Therefore in order to prove that Alg$\mathcal{L}$ is a von Neumann algebra, it is sufficient to show that Alg$\mathcal{L}$ is self-adjoint.

2. Examples of Alg$\mathcal{L}$

EXAMPLE 1. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \dots \}$ and let $\mathcal{L}$ be the lattice generated by $\mathcal{F} = \{ [e_1, e_2], [e_3, e_4], [e_5, e_6], \dots \}$ Then $\vee\mathcal{F} = \mathcal{H}$ and Alg$\mathcal{L}$ consists of matrices of the following form.

$$\begin{pmatrix} * & * & & & \\ * & * & & & \\ & & * & * & \\ & & * & * & \\ & & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \dots \}$, where all non-starred entries are zeros. Since Alg$\mathcal{L}$ is self-adjoint, Alg$\mathcal{L}$ is a von Neumann algebra

EXAMPLE 2. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \dots \}$ and let $\mathcal{F} = \{ [e_1, e_2], [e_3, e_4, e_5], [e_6, e_7, e_8, e_9], \dots \}$. Let $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. Since Alg$\mathcal{L}$ consists of matrices of the following form:

$$\begin{pmatrix}
* & * & & & & & & & & \\
* & * & & & & & & & & \\
& & * & * & * & & & & & \\
& & * & * & * & & & & & \\
& & * & * & * & & & & & \\
& & & & & * & * & * & * & \\
& & & & & * & * & * & * & \\
& & & & & * & * & * & * & \\
& & & & & * & * & * & * & \\
& & & & & & & & & \ddots
\end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \dots \}$, where all non-starred entries are zeros,

Alg$\mathcal{L}$ is a von Neumann algebra and Alg$\mathcal{L}$ = Alg$\mathcal{F}$.

EXAMPLE 3. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_i : i = 1, 2, \dots \}$ and let $\mathcal{L}_\infty$ be the subspace lattice generated by $\mathcal{F} = \{ [e_{2i-1}], [e_{2i-1}, e_{2i}, e_{2i+1}] : i = 1, 2, \dots \}$. Then $\vee\mathcal{F} = \mathcal{H}$ and Alg$\mathcal{L}_\infty$ consists of matrices of the following form :

$$\begin{pmatrix}
* & * & & & \\
& * & & & \\
& * & * & * & \\
& & & * & \\
& & & * & \\
& & & & \ddots
\end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \dots \}$, where all non-starred entries are zeros. Because Alg$\mathcal{L}_\infty$ is not self-adjoint, Alg$\mathcal{L}_\infty$ is not a von Neumann algebra. We know that Alg$\mathcal{L}_\infty$ is a tridiagonal algebra

EXAMPLE 4. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \dots \}$ and let $\mathcal{F} = \{ [e_i] : i = 1, 2, \dots \}$ and let $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. If A is in Alg$\mathcal{L}$, then A is the matrix which has the form :

$$\begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \dots \}$, where all non-starred entries are zeros. Hence Alg$\mathcal{L}$ is a von Neumann algebra. Let $\{E_i\}$ be a subset of $\mathcal{L}$, where E_i is the orthogonal projection from $\mathcal{H}$ onto $[e_1, e_2, \dots, e_i]$. Then $\{ E_i \}$ converges strongly to I. Since I is not in $\mathcal{L}$ and E_i is in $\mathcal{L}$ $(i = 1, 2, \dots)$, $\mathcal{L}$ is not strongly closed In particular, $\mathcal{L}$ is not complete.

EXAMPLE 5. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \dots \}$. Let $\mathcal{F} = \{ [e_1, e_2], [e_3, e_4], [e_5, e_6], \dots \}$ and $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. Then $\vee\mathcal{F} = \mathcal{H}$ and Alg$\mathcal{L}$ consists of matrices of the following form:

$$\begin{pmatrix} * & * & & & \\ * & * & & & \\ & & * & * & \\ & & * & * & \\ & & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \dots \}$, where all non-starred entries are zeros. If $\mathcal{L}_1$ is the lattice generated by $\{ [e_i] : i = 1, 2, \dots \}$, then $\mathcal{L}$ is a proper subset of $\mathcal{L}_1$ and Alg$\mathcal{L}_1$ consists of matrices of the following form :

$$\begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \dots \}$, where all non-starred entries are zeros. Thus Alg$\mathcal{L}_1$ is a proper subset of Alg$\mathcal{L}$.

EXAMPLE 6. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \ldots \}$ and let $\mathcal{F} = \{ [e_i] : i = 1, 2, \ldots \}$. Suppose that $\mathcal{L}$ is the complete lattice generated by $\mathcal{F}$. Then there is not a family $\mathcal{G}$ of mutually orthogonal projections in $\mathcal{L}$ which generates $\mathcal{L}$.

3. General Theorems

LEMMA 1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be families of orthogonal projections acting on a Hilbert space $\mathcal{H}$. If $\mathcal{L}_1 \subset \mathcal{L}_2$, then $Alg\mathcal{L}_2 \subset Alg\mathcal{L}_1$.*

PROOF. Let A be in Alg$\mathcal{L}_2$. Then $AE = EAE$ for all E in $\mathcal{L}_2$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$, $AE = EAE$ for all E in $\mathcal{L}_1$. Hence A is in Alg$\mathcal{L}_1$.

LEMMA 2 ([6]). *Let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then $Alg\mathcal{L} = Alg\mathcal{F}$.*

THEOREM 3. *Let $\mathcal{H}$ be a separable Hilbert space let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then $Alg\mathcal{L}$ is a von Neumann algebra.*

PROOF. It is sufficient to show that Alg$\mathcal{L}$ is self-adjoint. Let A be an element in Alg$\mathcal{L}$. Suppose that $\mathcal{F} = \{ E_1, E_2, \ldots \}$, where E_i is an orthogonal projection acting on $\mathcal{H}$ for all $i = 1, 2, \ldots$. Since A is in Alg$\mathcal{L}$, $AE_i = E_iAE_i$ for all $i = 1, 2, \ldots$. Since $AE_i^{\perp} = E_i^{\perp}AE_i^{\perp}$ for all $i = 1, 2, \ldots$, and hence $E_i^{\perp}A^* = E_i^{\perp}A^*E_i^{\perp}$ for all $i = 1, 2, \ldots$ Since $E_i^{\perp} = I - E_i$ for each $i = 1, 2, \ldots$,

$$\begin{aligned} E_i^{\perp}A^* &= A^* - E_iA^* \\ &= (I - E_i)A^*(I - E_i) \\ &= A^* - E_iA^* - A^*E_i + E_iA^*E_i. \end{aligned}$$

Hence $A^*E_i = E_iA^*E_i$ for all $i = 1, 2, \ldots$. Therefore by Lemma 2, A^* is in Alg$\mathcal{L}$, i.e. Alg$\mathcal{L}$ is self-adjoint.

LEMMA 4 ([2]). *Let $\mathcal{H}$ be a separable Hilbert space let $\mathcal{L}$ be a lattice of orthogonal projections acting on $\mathcal{H}$. If $\mathcal{L}$ is strongly closed, then $\mathcal{L}$ is complete.*

The converse of Lemma 4 is not true in general.

LEMMA 5 ([3]). *Let $\mathcal{H}$ be a Hilbert space let $\mathcal{L}$ be a complete lattice of orthogonal projections acting on $\mathcal{H}$ If $\mathcal{L}$ need not be strongly closed.*

LEMMA 6 ([2]). *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{L}$ is a commutative complete lattice of orthogonal projections acting on $\mathcal{H}$, then $\mathcal{L}$ is strongly closed.*

LEMMA 7 *Let $\mathcal{H}$ be a separable Hilbert space let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$. Let $\mathcal{L}$ be the complete lattice generated by $\mathcal{F}$. Then $\mathcal{L}$ is commutative.*

LEMMA 8 ([2]). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative lattice of orthogonal projections acting on $\mathcal{H}$. Then the strong closure of $\mathcal{L}$ is itself a lattice.*

LEMMA 9. *Let $\mathcal{F}$ be a family of mutually orthogonal projections acting on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. If $\mathcal{L}_1$ is the complete lattice generated by $\mathcal{F}$. then $\mathcal{L}_1$ is the strong closure of $\mathcal{L}$ and $Alg\mathcal{L} = Alg\mathcal{L}_1$.*

PROOF. Since $\mathcal{F}$ is a family of mutually orthogonal projections acting on a $\mathcal{H}$, $\mathcal{F}$ is commutative and therefore $\mathcal{L}$ is commutative. By Lemma 8, the strong closure of $\mathcal{L}$ is a lattice and by Lemma 4 the strong closure of $\mathcal{L}$ is complete. By Lemmas 6 and 7, $\mathcal{L}_1$ is strongly closed. Hence $\mathcal{L}_1$ contains the strong closure of $\mathcal{L}$. Therefore $\mathcal{L}_1$ is the strong closure of $\mathcal{L}$. By Lemma 1, Alg$\mathcal{L}_1$ is contained in Alg$\mathcal{L}$. Let A be in Alg$\mathcal{L}$ and E be in $\mathcal{L}_1$. Then there exists a family $\{E_\alpha\}$ of $\mathcal{L}$ such that $\{E_\alpha\}$ converges strongly to E. Since $\{E_\alpha\}$ converges strongly to E, $\{AE_\alpha\}$ and $\{E_\alpha AE_\alpha\}$ converge strongly to AE and EAE, respectively. Since $AE_\alpha = E_\alpha AE_\alpha$ for each α, $AE = EAE$. Therefore A is in Alg$\mathcal{L}_1$

If E and F are orthogonal projections from a Hilbert space $\mathcal{H}$ onto subspaces Y and Z, respectively, the following conditions are equivalent ([6]) :

(a) $X \subset Y$,
(b) $FE = E$,
(c) $EF = E$,
(d) $E \leq F$.

THEOREM 10. *Let $\mathcal{L}$ be a lattice of orthogonal projections acting on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{F} = \{F : F$ is a nonzero minimal element in $\mathcal{L}\}$. Then $\mathcal{F}$ is a mutually orthogonal family.*

PROOF. Let E and F be elements of $\mathcal{F}$. Suppose that $E \wedge F \neq 0$ and $E \neq F$. Since $E \wedge F \leq F$ and F is minimal, $E \wedge F = F$. Hence $F \leq E$. Since E and F are minimal, $E = F$. So $E \wedge F = 0$ or $E = F$. Hence $\mathcal{F}$ is a mutually orthogonal family.

THEOREM 11. *Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then $Alg\mathcal{L}$ is a von Neumann algebra if and only if $Alg\mathcal{L} = \mathcal{L}'$.*

PROOF. Necessity : If A is in $\mathcal{L}'$, then $AE = EA$ for all E in $\mathcal{L}$ Since $AE = AEE = EAE$ for all E in $\mathcal{L}$, A is in Alg$\mathcal{L}$. Since Alg$\mathcal{L}$ is a von Neumann algebra, A^* is in Alg$\mathcal{L}$ for all A in Alg$\mathcal{L}$. If A is in Alg$\mathcal{L}$, then $AE = EAE$ and $A^*E = EA^*E$ for all E in Alg$\mathcal{L}$. Hence $AE = EA$ for all E in Alg$\mathcal{L}$. Thus A is in $\mathcal{L}'$.

Sufficiency : It is sufficient to show that Alg$\mathcal{L}$ is self-adjoint. Suppose that A is in Alg$\mathcal{L}$. Since Alg$\mathcal{L} = \mathcal{L}'$, $AE = EAE$ and $AE = EA$ for all E in $\mathcal{L}$. Hence for all E in $\mathcal{L}$ $EA = EAE$, that is, $A^*E = EA^*E$ for all E in $\mathcal{L}$. Therefore A^* is in Alg$\mathcal{L}$

LEMMA 12 ([2]). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice of orthogonal projections acting on $\mathcal{H}$. Then $\mathcal{L}$ is reflexive.*

LEMMA 13. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}$ be a complete lattice of orthogonal projections acting on $\mathcal{H}$. Let $\mathcal{F} = \{F :$ F is a nonzero minimal element in $\mathcal{L}\}$. If E is a nonzero element in $\mathcal{L}$, then there exists E_0 in $\mathcal{F}$ such that $E_0 \leq E$.*

PROOF. Since E is a nonzero element in $\mathcal{L}$, E contains a unit vector e of $\mathcal{H}$. Let $\mathcal{S} = \{E_i \in \mathcal{L} : E_i \leq E,\ e \in E_i\}$. Then $\mathcal{S}$ is a nonempty set and it is a partially ordered set by set inclusion $\subset$. In order to apply the Zorn's Lemma we must show that every chain in $\mathcal{S}$ has a lower bound in $\mathcal{S}$. Let $\mathcal{M} = \{C_i : i \in \Gamma\}$ be a chain in $\mathcal{S}$ and $C = \wedge_{i\in\Gamma} C_i$.

Then $e \in C$ and $C \leq E$. Since $\mathcal{L}$ is complete, C is in $\mathcal{S}$, C is a lower bound in $\mathcal{S}$. Therefore there exists a minimal element E_0 in $\mathcal{S}$.

THEOREM 14. *Let $\mathcal{L}$ be commutative subspace lattice of orthogonal projections acting on a separable Hilbert space $\mathcal{H}$. If Alg$\mathcal{L}$ is a von Neumann algebra, then there exists a family $\mathcal{F}$ of mutually orthogonal projections in $\mathcal{L}$ which generates completely $\mathcal{L}$.*

PROOF. Let $\mathcal{F} = \{F : F \text{ is a nonzero minimal element in } \mathcal{L}\}$. Then $\mathcal{F}$ is a mutually orthogonal family by Theorem 10. We shall show that $\mathcal{L} = G(\mathcal{F})$, where $G(\mathcal{F})$ is the complete lattice generated by $\mathcal{F}$. Let E be a nonzero element in $\mathcal{L}$. Suppose that E is not in $G(\mathcal{F})$. If $EF = 0$ for all F in $\mathcal{F}$, then $E(\vee\mathcal{F}) = 0$. Since E is in $\mathcal{L}$, there exists an element E_0 in $\mathcal{F}$ such that $E_0 \leq E$ by Lemma 13. Since $E_0(\vee\mathcal{F}) \neq 0$, $E(\vee\mathcal{F}) \neq 0$. It is a contradiction.

Suppose that $EF \neq 0$ for some F in $\mathcal{F}$. Since $E \wedge F$ is in $\mathcal{L}$ and F is minimal, $E \wedge F = F$. Hence $F \leq E$. Put $\mathcal{F}_1 = \{D \in \mathcal{F} . D \leq E\}$. Then $\vee\mathcal{F}_1 \leq E$. Suppose that $\vee\mathcal{F}_1 = E$. Since $G(\mathcal{F})$ is complete, $\vee\mathcal{F}_1$ is in $G(\mathcal{F})$. It is a contradiction.

If $\vee\mathcal{F}_1$ is a proper subprojection of E, then $E - \vee\mathcal{F}_1$ is in $\mathcal{L}$. For each A in Alg$\mathcal{L}$, by Theorem 11

$$\begin{aligned} A(E - \vee\mathcal{F}_1) &= AE - A(\vee\mathcal{F}_1) \\ &= EA - (\vee\mathcal{F}_1)A \\ &= (E - \vee\mathcal{F}_1)A. \end{aligned}$$

Hence $E - \vee\mathcal{F}_1$ is in LatAlg$\mathcal{L}$ Since $\mathcal{L}$ = LatAlg$\mathcal{L}$ by Lemma 12, $E - \vee\mathcal{F}_1$ is in $\mathcal{L}$. By Lemma 13, there exists a nonzero minimal element E_1 in $\mathcal{L}$ such that $E_1 \leq E - \vee\mathcal{F}_1$. It is a contradiction. Since $\mathcal{L}$ contains $G(\mathcal{F})$, $\mathcal{L} = G(\mathcal{F})$.

THEOREM 15([8]). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. $\mathcal{V}$ is of type I if and only if any nonzero orthogonal projection in $\mathcal{V}$ contains an abelian nonzero orthogonal projection*

LEMMA 16([9]). *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is of type I.*

LEMMA 17([10]). *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{V}$ is finite if and only if $\dim\mathcal{H} < \infty$.*

LEMMA 18. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space with an orthonormal basis $\{ e_1, e_2, \ldots \}$ and let $\mathcal{F} = \{ [e_i] : i = 1, 2, \ldots \}$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then $Alg\mathcal{L}$ is of type I_∞.*

PROOF. By Lemma 2, Alg$\mathcal{L}$ = Alg$\mathcal{F}$. Let A be in Alg$\mathcal{L}$. Then A is diagonal. If E is a nonzero orthogonal projection in Alg$\mathcal{L}$, then each diagonal element of E is 0 or 1. So E contains a nonzero abelian orthogonal projection E_{ii} in Alg$\mathcal{L}$ for some i, where E_{ii} is the matrix whose (i,i)-component is 1 and all other components are 0. By Theorem 15, Alg$\mathcal{L}$ is of type I and hence Alg$\mathcal{L}$ is of type I_∞ by Lemma 17.

THEOREM 19. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then $Alg\mathcal{L}$ is of type I_∞.*

PROOF. Suppose that $\mathcal{F} = \{ E_1, E_2, \ldots \}$ and $\mathcal{H}_i$ is the subspace of $\mathcal{H}$ such that $E_i(\mathcal{H}) = \mathcal{H}_i$ for all $i = 1, 2, \ldots$ Let A be in Alg$\mathcal{L}$. Since Alg$\mathcal{L}$ = Alg$\mathcal{F}$ by Lemma 2, A is in Alg$\mathcal{F}$. Hence A has the following matrix form on $\oplus_{i=1}^\infty \mathcal{H}_i$

$$\begin{array}{c} \begin{array}{cccc} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \cdots \end{array} \\ \begin{pmatrix} A_{11} & & & \\ & A_{22} & & \mathbf{0} \\ & & A_{33} & \\ & \mathbf{0} & & \ddots \end{pmatrix} \end{array}$$

where $A_{ii} : \mathcal{H}_i \to \mathcal{H}_i$ is the operator such that $A_{ii} = A\mid_{\mathcal{H}_i}$ for all $i = 1, 2, \ldots$.

Let E be a nonzero orthogonal projection in Alg$\mathcal{L}$. Then E has the following matrix form on $\oplus_{i=1}^{\infty}\mathcal{H}_i$:

$$\begin{pmatrix} E_{11} & 0 & 0 & 0 & \cdots \\ 0 & E_{22} & 0 & 0 & \cdots \\ 0 & 0 & E_{33} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \\ & & & & \end{pmatrix}$$

where E_{ii} is the orthogonal projection acting on $\mathcal{H}_i$ such that $E_{ii} = E\mid_{\mathcal{H}_i}$ for all $i = 1,2, \ldots$ and E_{kk} is nonzero for some k. If E_{kk} is a nonzero orthogonal projection acting on $\mathcal{H}_k$ for some k, E_{kk} contains a subprojection F_{kk} of rank one. Let F be the orthogonal projection acting on $\oplus_{i=1}^{\infty}\mathcal{H}_i$ such that $E_kF\mid_{\mathcal{H}_k} = F_{kk}$ and $E_iF\mid_{\mathcal{H}_j} = 0$ if $i \neq k$ or $j \neq k$. Then F is in Alg$\mathcal{L}$ and F is a nonzero abelian subprojection of E. Hence Alg$\mathcal{L}$ is of type I by Theorem 15 By Lemma 17, Alg$\mathcal{L}$ is of type I_∞.

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