SOME THEOREMS ON RECURRENT FINSLER SPACES BY THE PROJECTIVE CHANGE

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ABSTRACT If any geodesic on $F^n$ is also a geodesic on $\tilde{F}^n$ and the inverse is true, the change $\sigma : L \rightarrow \tilde{L}$ of the metric is called projective. In this paper, we will find the condition that a recurrent Finsler space remains to be a recurrent one under the projective change.

0. Introduction

It is known that the Douglas tensor and the Weyl tensor are invariant under any projective change. Moreover, $h$-curvature tensor in the Berwald connection $BF$ is also invariant under a special projective change (Z-projective change). In the paper ([4]), M. Fukui and T. Yamada dealt with it and obtained some results. A Finsler space of zero curvature remains a space of zero curvature by the Z-projective change which is characterized as $Q_i = 0$.

In the paper ([3]), Bácsó, Illesvay and Kis investigated Finsler space $F^n$ and $\tilde{F}^n$ in which the $h(hv)$-torsion tensors coincide, that is, $\tilde{C}_{ijk} = C_{ijk}$. And they gave an example for this kind of spaces. Further, if the projective factor $p$ satisfies $p_{ij} = 0$ ([4]), the $hv$-curvature tensor $G$ is invariant under the projective change.

In this paper, we are devoted to $H$-recurrent space, $C$-recurrent space and $G$-recurrent space under the projective change.
1. A Berwald connection

Let $F^n = (M^n, L)$ be an $n$-dimensional Finsler space, where $M^n$ is a connected differential manifold of dimension $n$ and $L(x, y)$ is the fundamental function defined on the manifold $T(M)/0$ of none-zero tangent vectors. We assume that $L$ is positive and the fundamental metric tensor $g_{ij} = (1/2)\hat{\partial}_j \hat{\partial}_i L^2$ is positive definite, where $\hat{\partial}_i = \partial / \partial y^i$.

A geodesic on $F^n$ is given by the differential equation

$$d^2 x^i / ds^2 + 2G^r_i (\hat{\partial}_r x^i) = 0,$$

where $s$ is the arc-length of the curve. In the present paper, we are mainly concerned with the Berwald connection $B \Gamma = (G^i_j, G^i_j, 0)$, which is defined as: $G^r_j = \hat{\partial}_j G^r$, $G^r_j k = \partial_k G^r j$. For a Finsler tensor field $X^i$, the $h$-covariant derivative with respect to $B \Gamma$ is given by

$$(1.1) \quad X^h : i = \partial_i X^h - G^r_i (\hat{\partial}_r X^h) + X^r G^h r_i,$$

where $\partial_i = \partial / \partial y^i$.

For $B \Gamma$ we consider the torsion and curvatures. According to the theory of Finsler connection ([1],[5]), the $(v)$-torsion $R^1$ is the same with that of Cartan connection $C \Gamma$, because $B \Gamma$ and $C \Gamma$ have the common spray connection $(G^r_j)$. And the $h$-curvature tensor $R^2$ and the $hv$-curvature tensor $P^2$ are usually written as $H = (H^i_j k)$ and $G = (G^i_j k)$ respectively. These tensors are written as

$$(1.2) \quad H^i_j k = U_{(jk)} \{\partial_k G^i_j - G^r_i (\hat{\partial}_r G^j_k) + G^r_j G^i_r k\},
\quad G^i_j k = \hat{\partial}_h G^i_j k,$$

where $U_{(jk)}$ means the interchange of indices $j, k$ and subtraction.

Throughout the index 0 denotes the transvection by $y^i$ ([1]), for example, $y^i F^h i = F^h 0$. For later use, we introduce the following relations ([8]):

$$(1.3) \quad (a) \quad H^0 j k = H^i j k, \quad (b) \quad H^0 i k = H^i k, \quad (c) \quad H^i j k = -H^k j i,
\quad (d) \quad H^i j k = (1/3) U_{(jk)} \{\hat{\partial}_j H^i k\}, \quad (e) \quad H^k j k = \hat{\partial}_h H^i j k.$$
2. Projective changes of metrics

We consider two Finsler spaces $F^n = (M^n, L)$ and $F^n = (M^n, \bar{L})$ on a common underlying manifold $M^n$. Let the change $\sigma : L \rightarrow \bar{L}$ be a projective. It is well known that $\sigma$ is projective, if and only if there exists a $(1)p$-homogeneous Finsler scalar field $p(x, y)$ on $M^n$ satisfying

\begin{equation}
\bar{G}^i = G^i + py^i, \quad p \neq 0,
\end{equation}

at any $(x, y)$. This $p$ is called the projective factor.

We shall see how the torsion and curvature tensors are changed by a projective change. Let $BT = (G^i_j, \bar{G}^i_j, 0)$ be the Berwald connection on the space $\bar{F}^n = (M^n, \bar{L})$ obtained from $F^n = (M^n, L)$ by the projective change $\sigma$. Then, (2.1) immediately gives

\begin{equation}
\bar{G}^i_j = G^i_j + y^i_j p_j + \delta^i_j p,
\end{equation}

\begin{equation}
\bar{G}^i_j k = G^i_j k + y^i_j p_k + \delta^i_j p_k + \delta^k_j p_i,
\end{equation}

where we put $p_i = \hat{\partial}_ip$ and $p_{ij} = \hat{\partial}_ip_j$.

On the other hand, the $h\nu$-curvature tensor and the $h$-curvature tensor are given by

\begin{equation}
\bar{G}^h_{ij} k = G^h_{ij} k + y^h_{ij} p_{jk} + A_{(ij)k}\{\delta^h_{jk} p_k\},
\end{equation}

\begin{equation}
H^h_{ij} k = H^h_{ij} k + y^h_{ij} Q_{kj} + \delta^h_{k} Q_{ij} + U_{(ij)k}\{\delta^h_{jk} Q_k\},
\end{equation}

where we put $k = \partial_k, Q_i = p_{ij} - pp_i, Q_{ij} = U_{(ij)}p_{ij}, p_{ijk} = \hat{\partial}_i \hat{\partial}_j \hat{\partial}_k p$ and $A_{(ij)k}$ means cyclic permutation of the indices $i, j, k$ and summation. If $Q_i = 0$, from (2.4) the $h$-curvature tensor $H$ is invariant under the projective change. In the paper ([7]), S.C. Rastogi discussed the properties of the projective factor $p(x, y)$ satisfying the condition $Q_i = 0$. A projective change of a Finsler space of zero curvature is also a Finsler space of zero curvature if and only if the projective factor $p$ satisfies the equation $Q_i = 0$. 
Definition 2.1. ([4]) A projective change $\sigma$ is called a $\mathcal{Z}$-projective change if $Q_1 = 0$.

S.C. Rastogi ([7]) proved the following

Lemma 2.1. If $Q_1 = 0$, then the scalar $p(x, y)$ and its derivative satisfy the equations:

\[(2.5) \quad (a) \quad p_r H_j^{\tau_1} = 0, \quad (b) \quad A_{(1j)k}(prk H_j^{\tau_1}) = 0.\]

3. A $\mathcal{H}$-recurrent space

In the paper ([2]), S. Bácsó defined an $A$-recurrent Finsler space that is, for a tensor $A'_{k} = H'_{k} - H h'_{k}$,

\[(3.1) \quad A'_{k,0} = \psi(x, y) A'_{k},\]

where $\psi(x, y)$ is a positively homogeneous function of degree one in $x$ and $h'_{k}$ is an angular metric tensor. Similarly we introduce $\mathcal{H}$-recurrent space as follows:

Definition 3.1. A Finsler space is called a $\mathcal{H}$-recurrent space ([6]) if its $h$-curvature tensor satisfies the relation

\[(3.2) \quad H_{h'_{j}k,m} = \phi(x, y) H_{h'_{j}k},\]

where $\phi(x, y)$ is a positively homogeneous function of degree one in $y$ ([1]).

Meher's paper ([6]) was concerned with a symmetric Finsler space and obtained a relation of the Berwald's scalar curvature. Moreover he discussed a scalar function, which gives rise to the projective motion. A symmetric Finsler space is characterized by $H_{h'_{j}k;m} = 0$. Therefore a symmetric space is a $\mathcal{H}$-recurrent space with $\phi = 0$.

Let $B\Gamma$ be the Berwald connection on the space $\tilde{F}^n$ obtained from $F^n$. Then, from (1.1) the covariant derivative of the $h$-curvature tensor in $\tilde{F}^n$ is given by

\[(3.3) \quad \tilde{H}_{h'_{j}k;m} = \delta_{m} \tilde{H}_{h'_{j}k} - \delta_{a} \tilde{H}_{h'_{j}k} \tilde{G}^{a}_{m} + \tilde{H}_{h'_{j}k} \tilde{G}^{a}_{m} - \tilde{H}_{h'_{j}k} \tilde{G}^{a}_{m} - \tilde{H}_{h'_{j}k} \tilde{G}^{a}_{m} - \tilde{H}_{h'_{j}k} \tilde{G}^{a}_{m} - \tilde{H}_{h'_{j}k} \tilde{G}^{a}_{m}.\]
where $(;)$ denotes the $h$-covariant derivative with respect to $B\tilde{\Gamma}$. The $h$-curvature tensor is invariant under the $\mathcal{Z}$-projective change. Paying attention to (2.2), we get

\[ H^i_{jk;m} = H^i_{jk;m} + A_{(jk)m} \{ H^i_{jk}p_m \} - p\partial_m H^i_{jk} \]

\[ + H^a_{jk}p_am + H^a_{jk}\delta^i_m p_a - H^i_{jk}p_k \]

\[ - H^i_{jk}p_h - 3H^i_{jk}p_m + U_{(jk)} \{ H^i_{k0}p_jm \}. \]

Since $p(x, y)$ and $R^2(x, y)$ are homogeneous functions of degree one and zero in $y$ respectively, we find

\[ p_0 = p, \quad p_{m0} = 0, \quad \partial_0 H^i_{jk} = 0. \]

We assume that a $\mathcal{H}$-recurrent space $F^n$ is transformed into another $\mathcal{H}$-recurrent one $\tilde{F}^n$ by the $\mathcal{Z}$-projective change. And transvecting (3.4) with $y^m$ and $y^h$, from (1.3), (2.5) and (3.5) we have

\[ (\phi - \phi + 3p)H^i_{jk} + U_{(jk)} \{ H^i_{k0}p_j \} = 0. \]

Further, transvecting this with $y^k$, we obtain $(\phi - \phi + 4p)H^i_{k} = 0$, which implies $\phi = \phi + 4p$ by virtue of $H^i_{k} \neq 0$.

Summarizing up the above, we have the following

**THEOREM 3.1.** Let a $F^n$ and a $\tilde{F}^n$ be $\mathcal{H}$-recurrent spaces with the function $\phi$ and $\tilde{\phi}$ respectively. If a $F^n$ is transformed into a $\tilde{F}^n$ by the $\mathcal{Z}$-projective change, then we have the relation $\phi = \tilde{\phi} + 4p$, where $p$ is the projective factor.

**4. A $C$-recurrent space**

**DEFINITION 4.1** A Finsler space $F^n$ is called a $C$-recurrent if the $h(hv)$-torsion tensor $C$ satisfies the following condition

\[ C_{ijk,0} = \psi(x, y)C_{ijk}, \]

where $\psi(x, y)$ is a positively homogeneous function of degree one in $y$. 

\[ (4.1) \]
In the paper ([3]), authors discussed pairs of Finsler spaces $F^m$ and $ar{F}^m$ in which the $h(hv)$-torsion tensors coincide, that is,
\begin{equation}
\bar{C}_{ijk} = C_{ijk}.
\end{equation}
They also gave an example for this kind of spaces.

Let's assume that Finsler spaces $F^m$ and $\bar{F}^m$ which satisfy (4.2). In $B\bar{\Gamma}$ of $\bar{F}^m$, from (1.1) and (2.2) we have
\begin{equation}
C_{ijk,m} = C_{ijk,m} - p\hat{\partial}_m C_{ijk} - C_{ijk}p_m - \Omega_{(ijklm)}\{C_{ijk}p_m\},
\end{equation}
where $\Omega_{(ijklm)}$ means cyclic permutation of the indices $i, j, k, m$ and summation.

Let's consider the projective change $\sigma : L \rightarrow \bar{L}$, where $F^n$ is an arbitrary Finsler space but $\bar{F}^m$ is a $C$-recurrent Finsler space, that is,
\begin{equation}
\bar{C}_{ijk,0} = \bar{\psi}(x, y)\bar{C}_{ijk},
\end{equation}
where $(\bar{\cdot})$ denotes the $h$-covariant derivative in $B\bar{\Gamma}$.

Transvecting (4.5) with $y^m$, we obtain
\begin{equation}
C_{ijk,0} = (\bar{\psi} + p)C_{ijk}.
\end{equation}
Putting $\psi = \bar{\psi} + p$, we find that $F^n$ is also $\mathcal{W}$-recurrent.

Thus we have the following

**Theorem 4.1.** Let $F^n$ and $\bar{F}^m$ be two Finsler spaces which are related by the condition (4.2). If a Finsler space $F^n$ can be transformed into a $C$-recurrent Finsler $\bar{F}^m$ with the function $\bar{\psi}$ by the projective change, then $F^n$ is also a $C$-recurrent with the function $\psi = \bar{\psi} + p$.

A Finsler space is called a Landsberg space ([1]), if the Berwald connection $B\Gamma$ coincides with the Rund connection $R\Gamma$. It is well known that a Finsler space is a Landsberg space if and only if $C_{ijkl} = 0$ or $B\Gamma$ is $h$-metrical. Transvevecting (4.3) with $y^m$ and using the relation $g_{ijk} = -2C_{ijk,0}$, we get $pC_{ijk} = 0$, which implies that the space is a Riemannian by virtue of $p \neq 0$. Thus we have

**Corollary 4.2.** Let $F^n$ and $\bar{F}^m$ be two Finsler spaces which are related by the condition (4.2). If a Landsberg space $F^n$ can be transformed into another Landsberg space by the projective change, then the space is a Riemannian.
5. A $G$-recurrent space

Now we consider the case of the $hv$-curvature tensor, which is given by (2.3). A Finsler space is called a Berwald space, if the connection coefficients $G_{jk}^i$ of $B\Gamma$ are functions of position $x$ alone, in any coordinate system, that is, the $G_{jk}^i = 0$

Next, similarly to the $C$-recurrent case, we can define the $G$-recurrent space

**Definition 5.1.** A Finsler space $F^n$ is called $G$-recurrent if the $hv$-curvature tensor satisfies the following condition

\[(5.1)\]

\[G_{jk}^i = \varphi(x, y)G_{jk}^i,\]

where $\varphi(x, y)$ is a positively homogeneous function of degree one in $y$.

From (2.3), if $p_{ij} = 0$, the $hv$-curvature tensor is invariant. The projective change is called a $B$-projective ([4]) if $p_{ij} = 0$. Therefore, we can see that a Berwald space remains to be a Berwald one by the $B$-projective change. Since the $hv$-curvature tensor is a positively homogeneous function of degree -1 in $y$, we find $\hat{\partial}_0 G_{jk}^i = -1$. And it satisfies the identities ([1], [8]):

\[(5.2)\]

\[G_{0jk} = G_{h0}^i = G_{h0}^i = 0\]

We are concerned with the projective change $\sigma : L \rightarrow \tilde{L}$, where $F^n$ is an arbitrary but $\tilde{F}^n$ is $D$-recurrent. From (1.2) and (2.2) we get

\[(5.3)\]

\[G_{hjk} = G_{hjk}^i - p\hat{\partial}_m G_{hjk}^i + G_{hjk}^i + G_{hjk}^i \hat{\partial}_m \Delta^i - \hat{\partial}_m G_{hjk}^i = 0.\]

Suppose that the projective factor satisfies a condition $G_{hjk}^r p_r = 0$, which we denote by $G$-condition. Transvecting (5.3) with $y^m$ and taking account of $\hat{\partial}_0 G_{hjk}^i = -1$, we obtain

\[(5.4)\]

\[G_{hjk}^i = \varphi G_{hjk}^i.\]

Putting $\varphi = \varphi$, we find that $F^n$ is also $G$-recurrent.

Conversely, if $\tilde{F}^n$ and $F^n$ are $G$-recurrent spaces with the function $\varphi = \varphi$, then from (5.3) we get $G_{hjk}^r p_r = 0$.

Thus we have the following;
Theorem 5.1. If a Finsler space $F^n$ can be transformed into a $G$-recurrent Finsler space $\tilde{F}^n$ with the function $\tilde{\varphi}$ by the projective change, then $F^n$ must be $G$-recurrent one with the function $\varphi = \tilde{\varphi}$, if and only if the projective factor $p$ satisfies $G$-condition.

References


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