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A BASIS OF THE SPACE OF MEROMORPHIC DIFFERENTIALS ON RIEMANN SURFACES

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ABSTRACT. In this paper, we compute a basis of the space of meromorphic differentials on a Riemann surface, holomorphic away from two fixed points. This basis consists of the differentials which have the expected zero or pole order at the two fixed points.

0. Introduction

Let Γ be an arbitrary compact Riemann surface of genus $g \geq 2$ with two distinguished points $P_\pm$. In [4], it is proved that there exists a basis of the space of meromorphic vector fields on Γ, holomorphic away from two fixed points $P_\pm$. This basis is defined and uniquely determined up to a constant by the following behaviour near $P_\pm$:

$$e_i = a_i^\pm z_\pm^{\pm i - g_0 + 1}(1 + O(z_\pm))\frac{\partial}{\partial z_\pm},$$

where $g_0 = \frac{3}{2}g$, i takes on integral values ($i = \cdots, -1, 0, 1, \cdots$) for even g and half-integral values ($i = \cdots, -\frac{1}{2}, \frac{1}{2}, \cdots$) for odd g, and z_+ (resp. z_-) the local coordinate around P_+ (resp. P_-).

We carried out the same task, in [1], for the quadratic differentials. More precisely, we proved there exist a basis of the space of meromorphic quadratic differentials holomorphic away from the two points $P_\pm$, which is defined by the following behaviour near $P_\pm$:

$$E_i = c_i^\pm z_\pm^{\mp i + g_0 - 2}(1 + O(z_\pm))(dz_\pm)^2, \quad c_i^\pm \in \mathbb{C},$$

where g_0 and $z_\pm$ are the same as above.

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In this paper, we compute a basis of the space of meromorphic differentials on Γ, holomorphic away from the two fixed points $P_\pm$.

THEOREM 1. *There exist a basis of the space of differentials, holomorphic away from the two points $P_\pm$, which for $|i| > \frac{g}{2}$ is defined by the following behaviour near $P_\pm$:*

$$w_i = b_i^\pm z_\pm^{\mp i + \frac{g}{2} - 1}(1 + O(z_\pm))dz_\pm, \quad b_i^\pm \in \mathbb{C},$$

where i and $z_\pm$ are the same as above.

1. Proof of the Theorem 1

The group of isomorphy classes of line bundles over Γ is isomorphic to the group of linear equivalence classes of divisors. Using this fact, we can translate the results on the existence of meromorphic functions to the existence of holomorphic sections of certain line bundles. Let us just formulate one result:

$$\dim H^0(X, L) \begin{cases} = 0, & \deg L < 0 \\ \geq 1 - g + \deg L, & 0 \leq \deg L < 2g - 1 \\ = 1 - g + \deg L, & \deg L \geq 2g - 1\ , \end{cases}$$

where L is a holomorphic line bundle on Γ.

Suppose that P_i are points and n_i are integers ($i = 1, 2, \ldots, k$). We set

$$R = \bigotimes_i L_{P_i}^{\otimes n_i},$$

where L_{P_i} is the line bundle which has a section with exactly one zero at the point P_i and vanishes nowhere else. And we set

$$M = L \otimes R.$$

Then the space $H^0(\Gamma, M)$ of holomorphic sections of M is isomorphic to the space of meromorphic sections of the bundle L which are holomorphic outside the points P_i and have at most a pole of order n_i at the point P_i (have a zero of order at least $-n_i$ if $n_i < 0$. $i = 1, 2, \ldots, k$).

Let us take the bundles

$$M_i = \omega \otimes L_{P_+}^{i - \frac{g}{2} + 1} \otimes L_{P_-}^{-i - \frac{g}{2} + 1}.$$

Then

$$\begin{aligned}\deg M_i &= (2g-2)+(i-\frac{g}{2}+1)+(-i-\frac{g}{2}+1)\\ &= g<2g-1.\end{aligned}$$

By Riemann Roch theorem for line bundles

$$\dim H^0(\Gamma, M_i) \geq 1-g+\deg M_i = 1.$$

We will show that

$$\dim H^0(\Gamma, M_i) = 1,$$

and the corresponding meromorphic section of ω has exactly the required zero or pole order at $P_\pm$. Let $i-\frac{g}{2}+1=-n$, i.e., $-i-\frac{g}{2}+1=-g+2+n$.

Then

$$H^0(\Gamma, \mathcal{O}_{K-nP_+-(g-2-n)P_-}) \cong H^0(\Gamma, M_i)$$

and

$$\begin{aligned}\dim H^0(\Gamma, \mathcal{O}_{K-nP_+-(g-2-n)P_-}) &- \dim H^0(\Gamma, \mathcal{O}_{nP_++(g-2-n)P_-})\\ &= 1-g+\deg M_i = 1.\end{aligned}$$

Case 1. $n > g-1$.

If $n=g$, then

$$\dim H^0(\Gamma, \mathcal{O}_{K-gP_++2P_-}) - \dim H^0(\Gamma, \mathcal{O}_{gP_--2P_-}) = 1.$$

Let us consider the points which satisfy

$$\dim H^0(\Gamma, \mathcal{O}_{K-nP}) = g-n, \quad \text{for } \ g \geq n. \tag{1}$$

If (1) is not true for the point P, we call P a Weierstraβ point. Since there are only finitely many Weierstraβ points on Γ, we can avoid these points. So let $P_\pm$ be no Weierstraβ points. Then we get

$$\dim H^0(\Gamma, \mathcal{O}_{K-gP_+}) = 0,$$

hence

$$\begin{aligned}\dim H^0(\Gamma, \mathcal{O}_{gP_-}) &= -g+1+\deg(gP_+)\\ &= 1.\end{aligned}$$

Let k be the generator of this space. We can choose such a P_- that it is not a zero of k. So k is neither in $H^0(\Gamma, \mathcal{O}_{gP_--P_-})$ nor in $H^0(\Gamma, \mathcal{O}_{gP_--2P_-})$. This means that

$$\dim H^0(\Gamma, \mathcal{O}_{gP_--P_-}) = 0$$

and

$$\dim H^0(\Gamma, \mathcal{O}_{gP_+-2P_-}) = 0.$$

Now we get

$$\dim H^0(\Gamma, \mathcal{O}_{K-gP_++P_-}) = 0$$

and

$$\dim H^0(\Gamma, \mathcal{O}_{K-gP_++2P_-}) = 1.$$

Since $\dim H^0(\Gamma, \mathcal{O}_{K-gP_++P_-}) = 0$, the generator f_1 of $H^0(\Gamma, \mathcal{O}_{K-gP_++2P_-})$ has the right pole order at P_-. And since $\dim H^0(\Gamma, \mathcal{O}_{K-(g+1)P_+}) = 0$, we get

$$\dim H^0(\Gamma, \mathcal{O}_{(g+1)P_+}) = -g + 1 + g + 1 = 2.$$

We can choose a P_-(if it is necessary, we can change the point P_-) which has to satisfy

$$\dim H^0(\Gamma, \mathcal{O}_{(g+1)P_+-P_-}) = 1.$$

Then the dimension of $H^0(\Gamma, \mathcal{O}_{(g+1)P_+-2P_-})$ is 0 or 1.

If $\dim H^0(\Gamma, \mathcal{O}_{(g+1)P_+-2P_-}) = 1$, then there is a basis $\{g_1, g_2\}$ of $H^0(\Gamma, \mathcal{O}_{(g+1)P_+})$ which satisfies the following equations for nonzero a_1, a_2;

$$a_1 g_1(P_-) + a_2 g_2(P_-) = 0,$$

$$a_1 g_1'(P_-) + a_2 g_2'(P_-) = 0, \quad a_i \in \mathbb{C}\ .$$

This implies

$$(g_1 g_2' - g_1' g_2)(P_-) = 0.$$

Because there are finitely many points satisfying the equation above, we can avoid these points. So we always can get

$$\dim H^0(\Gamma, \mathcal{O}_{(g+1)P_+-2P_-}) = 0$$

by changing P_- suitably. This means that

$$\dim H^0(\Gamma, \mathcal{O}_{K-(g+1)P_++2P_-}) = 0,$$

so the generator f_1 has the right zero order at P_+.

Now consider the case $n = g + 1$. The proof for the general case goes by induction.

Since

$$\dim H^0(\Gamma, \mathcal{O}_{gP_- - 2P_-}) = 0,$$

we get

$$\dim H^0(\Gamma, \mathcal{O}_{gP_- - 3P_-}) = 0.$$

From Riemann-Roch

$$\dim H^0(\Gamma, \mathcal{O}_{K - gP_+ + 3P_-}) = -g + 1 + (2g - 2) - g + 3 = 2.$$

Let f_1 be as above and let f be a second element, such that $\{f_1, f\}$ is a basis of this vector space. We can solve the equation

$$af_1^{(g)}(P_+) + cf^{(g)}(P_+) = 0, \quad c \neq 0.$$

$f_2 = af_1 + cf$ is a vector such that $\{f_1, f_2\}$ is again a basis. f_2 has at least a zero of order $g + 1$ at P_+. We do not want f_2 to have a higher order zero. For this we have to make sure that

$$af_1^{(g+1)}(P_+) + cf^{(g+1)}(P_+) \neq 0$$

by choosing P_+ suitably. Now f_2 generates the subspace of

$$H^0(\Gamma, \mathcal{O}_{K-(g+1)P_+ + 3P_-}).$$

It has the right zero order at P_+. If we assume that it does not have the full pole order 3 at P_-, it would be an element of $H^0(\Gamma, \mathcal{O}_{K-gP_+ + 2P_-})$, hence it would be a multiple of f_1 which is a contradiction to its construction.

Case 2. $n < -1$.

If we change the role of P_+ and P_-, then we get case 2.

Therefore we get

$$\dim H^0(\Gamma, M_i) = 1$$

and the corresponding meromorphic section is given by differentials w_i with the following behaviour near $P_\pm$:

$$w_i = b_i^\pm z_\pm^{\mp i + \frac{g}{2} - 1}(1 + O(z_\pm))(dz_\pm), \quad b_i^\pm \in \mathbb{C}\,,$$

for $|i| > \frac{g}{2}$.

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