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THE EXISTENCE OF (n, r, t, k)-ARRANGEMENTS

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ABSTRACT. B. Grünbaum defined the arrangements of simple curves and got many combinatorial properties. In this paper, we studied the existence of (n, r, t, k)-arrangements and the existence of digon-free (n, r, t, k)-arrangements, which is a generalized version of Grünbaum's definition.

1. Introduction

When two curves meet at a point, they either intersect (cross) or osculate. If two curves intersect at a point, we call it an **intersection point**. If two curves osculate at a point, we call it a **kissing point** or an **osculation point**.

J. Malkevitch suggested a generalization of the arrangements of simple curves defined by B. Grünbaum [1, 3]. An **arrangement of n simple curves** in the Euclidean plane E^2 is a finite family of simple closed curves $\{C_1, C_2, \ldots, C_n\}$ with the following properties:

1) every pair of curves has exactly t intersection points (t is even) and exactly k kissing points in common,
2) exactly two curves meet at each point.

From this definition, it is obvious that every curve has the same number of points r, which is determined by the equation $r = (n-1)(t+k)$, $n \geq 2$. Hence, we will denote this arrangement by an $(\mathbf{n}, \mathbf{r}, \mathbf{t}, \mathbf{k})$-**arrangement of curves**.

Throughout this paper, we assume that every 4-tuple (n, r, t, k) consists of nonnegative integers satisfying the equation $r = (n-1)(t+k)$.

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Then $(n, r, 2, 0)$-arrangements without the second property are the arrangements of simple curves in Grünbaum's sense (see Figure 1).

The graph of an arrangement of simple curves is a plane graph each of whose vertices is either an intersection point or a kissing point, and whose edges are the segments of curves between each pair of adjacent points. We will use the term "point" and "vertex" interchangeably throughout this paper.

Let G be the graph of an (n, r, t, k)-arrangement. Then G is a plane 4-valent graph. If G does not have any faces which are digons, then such an arrangement is called **a digon-free (n, r, t, k)-arrangement of curves.**

Since all curves in an arrangement are simple, there is no $(1, r, t, k)$-arrangement. Also, every $(2, r, t, k)$-arrangement has digons. Thus we assume that $n \geq 3$, when we refer to a digon-free arrangement.

2. Existence of arrangements

Arrangements in Figure 1 illustrate the existence of an $(n, r, 2, 0)$-arrangement and a digon-free $(n, r, 2, 0)$-arrangement, respectively [cf. 1].

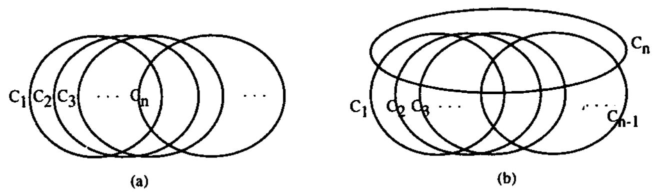


Figure 1. Examples of $(n, r, 2, 0)$-arrangement

For further study, let us define the operations O_1, O_2, and O_3 (Figure 2 (a), (b), and (c)). The operations O_1 and O_3 add two more intersection points, while the operation O_2 adds one more kissing point.

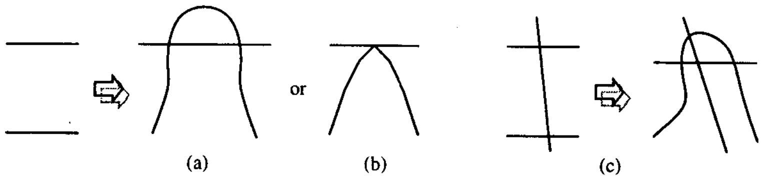


Figure 2. O_1, O_2, and O_3 operations

LEMMA 2.1. *If there is an (n, r, t, k)-arrangement with $t+k > 0$, then there exists an $(n, r', t+2m, k+k')$-arrangement where m and k' are positive integers and $r' = r + (n-1) \cdot (2m + k')$.*

PROOF. Let G be the graph of an (n, r, t, k)-arrangement with $t+k > 0$. Since $t + k > 0$, each pair of curves in G has at least one intersection point or kissing point. Let C_i and C_j be two different curves in G and let v be an intersection point or a kissing point (cf. Figure 3 (a), (b)). By applying the operation O_1 m times to the neighbor of the point v, we can increase the number of intersection points of the two curves to $t + 2m$. Similarly, by applying the operation O_2 k' times to the neighbor of the point v, we can increase the number of kissing points of the two curves to $k+k'$ as we wish that we would like (see Figure 3 (a), (b) for the case $m = 2, k' = 3$). Now, apply this method to every pair of curves, one by one. Then, the resulting graph is an $(n, r', t+2m, k+k')$-arrangement.□

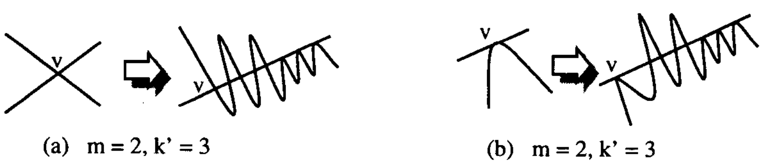


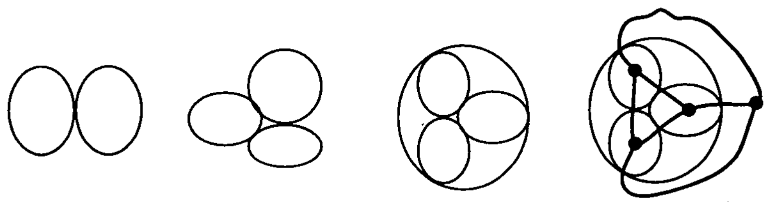
Figure 3. Adding intersection points and kissing points

LEMMA 2.2. *There is no $(n, r, 0, k)$-arrangement for $n \geq 5$.*

PROOF. We will proceed by induction on k. Suppose that $k = 1$. The arrangements in Figure 4 shows the existence of an $(n, r, 0, 1)$-arrangement for $n = 2, 3$ and 4. Now, suppose that there is an $(n, r, 0, 1)$-arrangement for some $n \geq 5$. Clearly each curve has only $(n-1)$ kissing points. Let C be a curve in this arrangement. If a curve resides inside the curve C (or outside the curve C, respectively), then all the other curves must reside inside C (or outside C, respectively). Otherwise, the curves inside C cannot kiss the curves that are outside of C without creating an intersection point, and this violates the condition $t = 0$.

Now, let us construct a new graph G' from the graph G of the given $(n, r, 0, 1)$-arrangement. Since every curve in an $(n, r, 0, 1)$-arrangement is simple, each curve separates the plane into two regions. Assign a vertex to each curve by adding and placing a vertex in the region that does not contain any other curves. And, join two vertices if the corresponding curves meet at a kissing point. The graph of thick lines in Figure 4 (d) illustrates this method. Then, the new graph G' is isomorphic to K_n since every curve osculates to all the other curves. We know that K_n is not planar if $n \geq 5$. Therefore, we cannot have an $(n, r, 0, 1)$-arrangement on the plane for all $n \geq 5$.

Suppose that our assertion is true for all $k < m$ and that there is an $(n, r, 0, m)$-arrangement. Let v be a kissing point of two curves C_i, C_j. Remove the vertex v and detach two curves at v, then the pair of curves C_i, C_j has $m - 1$ kissing points. By applying the same method to all the other pairs of curves, we have an $(n, r, 0, m - 1)$-arrangement and it contradicts our assumption. Therefore, there is no $(n, r, 0, k)$-arrangement if $n \geq 5$. □



(a) n = 2 (b) n=3 (c) n = 4 (d) a new graph

Figure 4.

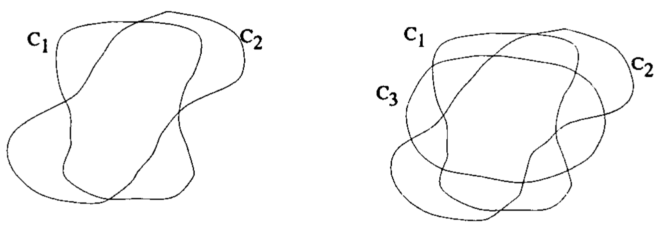
THEOREM 2.3. *For a 4-tuple (n, r, t, k), there is an (n, r, t, k)-arrangement, except for the case $n \geq 5$ and $t = 0$.*

PROOF. If $t = 0$, the result is clear by Lemma 2.2. Otherwise, there is an $(n, r', 2, 0)$-arrangement (Figure 1 (a)). Apply the operation O_1 $(t - 2)/2$ times and the operation O_2 k times to every pair of curves. The final arrangement is an (n, r, t, k)-arrangement. □

The arrangements described in Theorem 2.3 may not be a digon-free arrangement. However, a digon-free arrangement is more desirable because it can be a 3-polytopal graph. Hence, let us consider the construction of digon-free arrangements.

LEMMA 2.4. *There exists a digon-free* $(3, r, t, 0)$*-arrangement.*

PROOF. Consider a $(2, r', t, 0)$-arrangement which has a form like in Figure 5 (a). Now, draw the third curve that cuts the digons in the $(2, r', t, 0)$-arrangement to change them $2t$ triangles (Figure 5 (b)). The final arrangement is a digon-free $(3, r, t, 0)$-arrangement. □

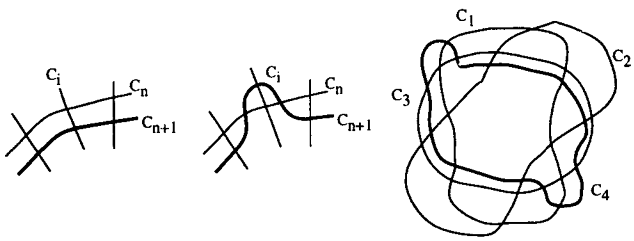


(a) a (2,4,4,0)-arrangement (b) a (3,8,4,0)-arrangement

Figure 5.

LEMMA 2.5. *Suppose that there exists a digon-free* $(n, r, t, 0)$*-arrangement. Then, there is a digon-free* $(n+1, r', t, 0)$*-arrangement where* $r' = r + t$.

PROOF. Let C be a digon-free $(n, r, t, 0)$-arrangement. Choose a simple curve C_n in C. Then, we can draw a new simple curve C_{n+1} such that C_{n+1} is parallel to the curve C_n and close enough not to have any points between C_n and C_{n+1} (Figure 6 (a)). Since C_n intersects every curve C_i $(i \leq n-1)$ t times, C_{n+1} also intersects all of these curves t times (except for the curve C_n). Now, pull over a part of the curve C_{n+1} along the other curve that cut through the curve C_n to make 2 intersection points (Figure 6 (b)). Repetition of this method at $\frac{t}{2}$ places yields the graph of an $(n+1, r', t, 0)$-arrangement (Figure 6 (c) shows an example for $n = 3, t = 4$). □



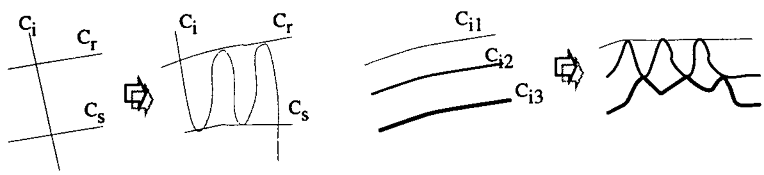
(a) drawing a parallel curve (b) making two intersection points (c) a (4,12,4,0)-arrangement

Figure 6.

As a result of Lemma 2.4 and Lemma 2.5, we have the following theorem.

THEOREM 2.6. *There exists a digon-free* $(n, r, t, 0)$*-arrangement for* $n \geq 3$.

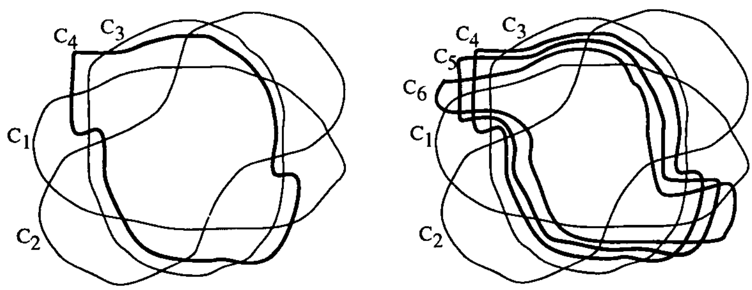
To demonstrate the existence of digon-free (n, r, t, k)-arrangements for an arbitrary number of kissing points k, we need to develop some methods to increase the number of kissing points in the arrangements. For this purpose, we will consider two operations K_1 and K_2. Suppose that a curve C_i intersects two curves C_r, C_s consecutively as in Figure 7 (a), then we change the curve C_i as in Figure 7 (a) to make k kissing points with the curve C_r and with the curve C_s. This operation is called K_1 operation. For the operation K_2, suppose that we have j curves $C_{i_1}, C_{i_2}, \ldots, C_{i_j}$ as in Figure 7 (b). By changing the curves as in Figure 7 (b), $j-1$ pairs of curves $C_{i_l}, C_{i_{l+1}}, l = 1, 2, \ldots, j-1$ have k kissing points each.



(a) K_1 operation for k = 2 (b) K_2 operation for k = 3

Figure 7. K_1 and K_2 operations

Now, let's construct a digon-free $(n, r, t, 0)$-arrangement according to the method described as above. We start from the digon-free (3,2t,t,0)-arrangement of the form in Figure 5 (b). First, draw the curve C_4 parallel to the curve C_3. Then there are t intersection points in the pair of curves $C_4, C_i, i = 1, 2$. Choose $t/2$ points in an alternating manner and pull a part of the curve C_4 out of the curve C_3 along the curve C_1 using the operation O_3. It produces t intersection points between the curves C_3 and C_4. Thus we have a digon-free $(4, 3t, t, 0)$-arrangement of curves (Figure 8 (a)). To construct a digon-free $(5, 4t, t, 0)$-arrangement of curves, draw the curve C_5 parallel to the curve C_4 and pull a part of the curve C_5 out of the curve C_4 along the curve C_1 at the same place where the curve C_4 was pulled out. By applying the same method again and again, we are able to construct a digon-free $(n, r, t, 0)$-arrangement (see Figure 8 (b) for the case $n = 6$).



(a) a (4,12, 4,0)-arrangement (b) a (6, 20, 4,0)-arrangement

Figure 8.

In the $(n, r, t, 0)$-arrangement constructed above, let's investigate the order of the curves which the curve $C_i, i = 1, 2, \cdots, n$ intersects. For example, the curve C_1 intersects the curves $C_2, C_3, C_4, \cdots, C_n, C_2, C_3, C_4, \cdots, C_n, C_2, C_3, C_4, \cdots$. Here is the table of the order of the curves that the curve C_i intersect. For convenience, we just use the index i instead of C_i.

$$1 : 2, 3, 4, 5, \cdots, n, 2, 3, 4, 5, \cdots, n, 2, \cdots$$
$$2 : 1, 3, 4, 5, \cdots, n, 1, n, \cdots, 5, 4, 3, 1, \cdots$$
$$3 : 1, 2, 4, 5, \cdots, n, 1, n, \cdots, 5, 4, 2, 1, \cdots$$
$$4 : 1, 2, 3, 5, \cdots, n, 1, n, \cdots, 5, 3, 2, 1, \cdots$$
$$5 : 1, 2, 3, 4, \cdots, n, 1, n, \cdots, 4, 3, 2, 1, \cdots$$
$$\vdots$$
$$n : 1, 2, 3, 4, \ldots, n-1, 1, n-1, \ldots, 4, 3, 2, 1, \ldots$$

As we see, the numbers are increasing at the beginning of each row. That is, we are able to apply operation K_1 to two pairs of curves $\{C_i, C_j\}, i \neq j$ and $\{C_i, C_{j+1}\}, i \neq j+1$ to increase the number of kissing points. Also this arrangement of curves has a configuration in Figure 7 (b), and the order of curves in this part is $3, 4, 5, \ldots, n$. Hence, it is possible to apply operation K_2 to any number of pairs of curves, $\{C_i, C_{i+1}\}, \{C_{i+1}, C_{i+2}\}, \ldots, \{C_{i+j-1}, C_{i+j}\}, 3 \leq i \leq n-2, 2 \leq j \leq n-i$.

Here is a good place to state our main theorem.

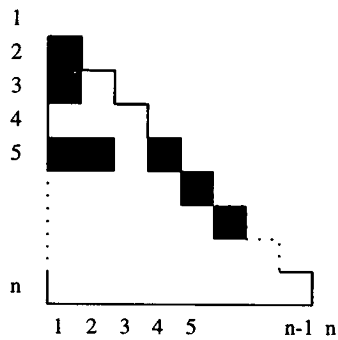
THEOREM 2.7. *There exists a digon-free (n, r, t, k)-arrangement except for the case $n \geq 5$ and $t = 0$.*

PROOF. We can exclude the case $n \geq 5$ and $t = 0$ by Lemma 2.2.

Since we have a digon-free $(n, r, t, 0)$-arrangement (Theorem 2.6), the only thing we have to do is to make k kissing points for each pair of curves. We do this by applying the operation K_1 and the operation K_2 appropriately.

Suppose that the given digon-free $(n, r, t, 0)$-arrangement was constructed in the special way as described above (cf. Figure 8 (b)). If $n = 3$, we can locate the configuration in Figure 7 (b) whose order is $1, 2, 3, 1$. By applying the operation K_2, we have a digon-free $(3, r, t, k)$-arrangement of curves.

For $n \geq 4$ curves, there are $\dfrac{n(n-1)}{2}$ pairs of curves. We will use the step like figure to display all the pairs of the curves (see Figure 9). We call it an $n-1$ stairs. In this figure, the cell in the (i, j)-position $(j < i)$ means the pair of two curves $\{C_i, C_j\}$. Then two pairs of curves $\{C_i, C_j\}$ and $\{C_i, C_{j+1}\}$ are either horizontally adjacent two cells $((i,j),(i,j+1)$ position) or vertically adjacent two cells $((j,i),(j+1,i)$ position). Furthermore j pairs of curves like $\{C_i, C_{i+1}\}, \{C_{i+1}, C_{i+2}\}, \ldots, \{C_{i+j-1}, C_{i+j}\}$, $2 \le j \le n-i$ are diagonally adjacent j cells. For example, see the blocks in black color in Figure 9. We call these blocks in black, a horizontal block, a vertical block, and a diagonal block of length j, respectively.



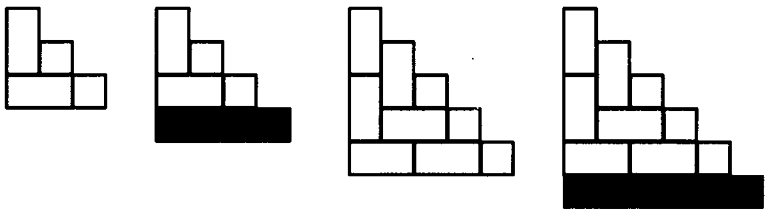
The n-1 stairs and 3 types of blocks

Figure 9. (n-1) stairs

For the pairs covered by either a horizontal block or a vertical block, we are able to apply the K_1 operation because there is a common curve C_i in two pairs which intersect the other two curves C_j, C_{j+1}. On the other hand, for the pairs in a diagonal block of length j, we can apply K_2 operation since there is a configuration of the curves in Figure 7 (b) whose order is $C_i, C_{i+1}, \ldots, C_{i+j}$. Thus to prove this theorem for n curves, it suffices to show that the $n-1$ stairs in Figure 9 can be covered by these 3 types of blocks.

Claim 1. Suppose that the $4m-1$ stairs and the $4m+1$ stairs are covered by the 3 types blocks, then the $4m$ stairs and $4m+2$ stairs are also covered by them.

The $4m$ stairs ($4m+2$ stairs, respectively) requires the addition of $4m$ cells ($4m+2$ cells, respectively) to the $4m-1$ stairs ($4m+1$ stairs, respectively) at the bottom. Furthermore, these $4m$ cells ($4m+2$ cells, resp.) can be covered by $2m$ ($2m+1$, resp.) horizontal blocks. Hence, if we cover the $4m-1$ stairs ($4m+1$ stairs, resp.) using 3 types blocks, then we are also able to cover $4m$ stairs ($4m+2$ stairs, resp.) too (see Figure 10 (a), (b) for the case $m=1$).



(a) 3 stairs and 4 stairs (b) 5 stairs and 6 stairs

Figure 10.

Claim 2. For $m \geq 4$, if the m stairs is covered by 3 types of blocks, then so is $m+4$ stairs.

Suppose that we covered the m stairs, then by adding $4 \times (m+1)$ cells covered by $2 \times (m+1)$ vertical blocks and the 3 stairs at the bottom of the k stairs, the $m+4$ stairs is also covered (Figure 11).

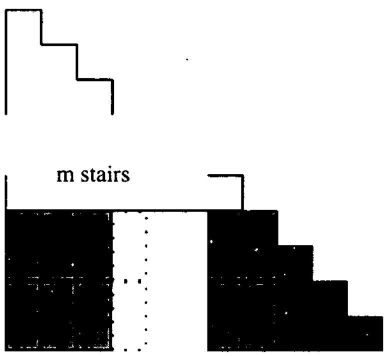


Figure 11.

Due to the Claim 1, if we can cover the 3 stairs and the 5 stairs, then the 4 stairs and the 6 stairs are also covered by 3 types of blocks (see Figure 10). Moreover, if we cover the m stairs for $m = 3, 4, 5, 6,$

then every m stairs is covered by 3 types blocks according to the Claim 2. That is, there is a way to change the given $(n, r, t, 0)$-arrangement $(n \geq 4)$ to (n, r, t, k)-arrangement by using the operation K_1 and the operation K_2. □

We can try to relate the (n, r, t, k)-arrangement to a block design. Let the points in the arrangement be varieties and let each simple curve be a block. Since each curve contains exactly r points, every block has r varieties and each variety appears exactly in two blocks. Some pairs of varieties appear in two blocks but not all the pairs. This is the only problem to relate the (n, r, t, k)-arrangements to a balanced incomplete block design. If we do not consider the λ condition, then there is a way to construct $(n, \frac{nr}{2}, 2, r, -)$ design by constructing a (n, r, t, k)-arrangement.

There are many ways to extend the area of the study on this kind of problem. First, if we change the word "exactly" in the first property in the definition to "at most", then we have the definition of **weak arrangements**. Second, changing the word "exactly two curves" in the second property in the definition to "exactly three curves" yields the 6-valent graphs. Finally, we only consider arrangements in Euclidean plane in this paper. If we allow arrangements in $\mathbf{E}^3$ space, then some facts in this paper are not true. For example, there is a $(5, 4, 0, 1)$-arrangement in $\mathbf{E}^3$ though there is none in $\mathbf{E}^2$ (see Lemma 2.2).

Also, there are some results of the Eberhard type problem for arrangements of simple curves [2].

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