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A NOTE ON FINITE CONDITIONS OF ORTHOMODULAR LATTICES

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ABSTRACT. We prove the following: every chain-finite OML is path-connected; every finite block of an OML L is path-connected with at least one other block in L; every OML with uniformly finite sites is path-connected.

1. Preliminaries

Several finite conditions of orthomodular lattices have been investigated [3, 6]. In this paper, we prove some properties of orthomodular lattices with some finite conditions.

An *orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies the *orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [5]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in B$.

A *subalgebra* of an OML L is a nonempty subset M of L which is closed under the operations $\vee$, $\wedge$ and $'$. We write $M \leq L$ if M is a subalgebra of L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the *relative interval sublattice* $M[a, b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the *relative orthocomplementation* $\sharp$ on $M[a, b]$ given by $c^\sharp = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a, b]$. In particular, $L[a, b]$ will be denoted by $[a, b]$ if there is no ambiguity.

The *commutator of* a *and* b of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. The set of all commutators of L is denoted by $ComL$ and L is said to be *commutator-finite* if $|ComL|$ is finite [2, 4]. For elements a, b of an OML, we say a *commutes with* b, in symbols $a \, \mathbf{C} \, b$, if $a * b = 0$. If M is a subset of an OML

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L, the set $\mathbf{C}(M) = \{x \in L \mid x\,\mathbf{C}\,m \quad \forall m \in M\}$ is called the *commutant* of M in L and the set $\mathbf{Cen}(M) = \mathbf{C}(M) \cap M$ is called the *center* of M. We note that $\mathbf{C}(L)$ is the center of L and $\mathbf{C}(L) = \bigcap\{\mathbf{C}(a) \mid a \in L\}$. An OML L is called *irreducible* if $\mathbf{C}(L) = \{0, 1\}$, and L is called *reducible* if it is not irreducible.

A *block* of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathfrak{A}_L$. Note that $\bigcup \mathfrak{A}_L = L$ and $\bigcap \mathfrak{A}_L = \mathbf{C}(L)$. An OML L is said to be *block-finite* if $|\mathfrak{A}_L|$ is finite.

For any e in an OML L, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called the *(principal) section generated by e.* Note that for $A, B \in \mathfrak{A}_L$, if $e \in A \cap B$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

Definition 1.1. For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A *path* in L is a finite sequence $B_0, B_1, \cdots, B_n$ $(n \geq 0)$ in $\mathfrak{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to *join* the blocks B_0 and B_n. The number n is said to be the *length* of the path. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. Using this element e, we say that A and B are *linked at e* (*strongly linked at e*) if $A \sim B$ $(A \approx B)$, and use the notation $A \sim_e B$ $(A \approx_e B)$. This element e is called a *vertex* of L and it is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1]. The set of all vertices of L is denoted by V_L and L is said to be *vertex-finite* if $|V_L|$ is finite.

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the section S_e*" to describe $A \overset{wk}{\sim} B$ [3].

Definition 1.2. Let L be an OML, and $A, B \in \mathfrak{A}_L$. We will say that A and B are *path-connected in L*, *strictly path-connected in L* if A and B are joined by a proper path, a strictly proper path, respectively. We will say A and B are *nonpath-connected* if there is no proper path joining A and B, and L is called *nonpath-connected* if there exist two blocks which are nonpath-connected. An OML L is called *path-connected in* L, *strictly path-connected in* L if any two blocks in L are joined by a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* iff each $[0, x]$ is path-connected for all $x \in L$.

Let L be an OML, and $A, B, C \in \mathfrak{A}_L$. If A and B are joined with a strictly proper path $A = B_0 \approx B_1 \approx \cdots \approx B_{m-1} \approx B_m = B$ and if B and C are joined with a strictly proper path $B = C_0 \approx C_1 \approx \cdots \approx C_{n-1} \approx C_n = C$ then A and C are strictly path-connected by the *concatenated path* $A = B_0 \approx B_1 \approx \cdots \approx B_{m-1} \approx B \approx C_1 \approx \cdots \approx C_{n-1} \approx C_n = C$.

The following propositions are known.

PROPOSITION 1.3. *Every finite direct product of path-connected OMLs is path-connected* [6].

PROPOSITION 1.4. *Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is nonpath-connected* [7].

2. Finite conditions of orthomodular lattices

We present some properties of orthomodular lattices with finite conditions. A *site* is a subalgebra of an OML L of the form $S = A \cap B$ where A and B are distinct blocks of L. An OML L is called *with uniformly finite sites* if and only if there exists a natural number n such that for all distinct blocks A, B of L, $|A \cap B| < n$. We prove the following: every chain-finite OML is path-connected; every finite block of an OML L is path-connected with at least one other block in L; every OML with uniformly finite sites is path-connected.

Greechie proved that every chain-finite OML is weakly path-connected [3]. We will prove a stronger result saying that every chain-finite OML is path-connected by using the following structure theorem of chain-finite OMLs.

PROPOSITION 2.1. *Every chain-finite OML L has a unique orthogonal* *decomposition* $L = L_0 \oplus L_1 \oplus \cdots \oplus L_n\,(0 \leq n)$ *where* L_0 *is a Boolean algebra and* $L_1, L_2, \cdots, L_n$ *are simple non-Boolean chain-finite OMLs.*

PROOF. Let L be a chain-finite OML. Then there exist only finitely many distinct minimal non-zero elements $c_1, c_2, \cdots, c_k \in \mathbf{C}(L)$, and $L = \bigoplus_{i=1}^{k}[0, c_i]$. Let $I = \{i \mid c_i \text{ is an atom of } L\}$, $L_0 = \bigoplus_{i \in I}[0, c_i]$ and $J = \{i \mid 1 \leq i \leq k,\ i \notin I\}$. Then $L = L_0 \oplus \bigoplus_{j \in J}[0, c_j]$ where L_0 is a Boolean algebra since each $[0, c_i] \subset \mathbf{C}(L)$ $(i \in I)$ and $[0, c_j]$ is an irreducible non-Boolean chain-finite OML for each $j \in J$. Let $n = |J|$. This completes the proof since every irreducible chain-finite OML is simple [5]. □

We prove that every chain-finite OML is path-connected using the following lemmas (2.2) and (2.3).

LEMMA 2.2. *Let* L *be an OML, and* $A, B \in \mathfrak{A}_L$. *If* $A \cap B = \mathbf{C}(L)$ *and* $A \cup B \not\leq L$, *then there exist* $C, D \in \mathfrak{A}_L$ *such that* $A \cap C \neq \mathbf{C}(L)$, $C \cap D \neq \mathbf{C}(L)$ *and* $D \cap B \neq \mathbf{C}(L)$.

PROOF. There exist c, d such that $c, d \in A \cup B$ and $c \vee d \notin A \cup B$ since $A \cup B \not\leq L$. Hence $c \vee d \notin \mathbf{C}(L) = \bigcap \mathfrak{A}_L$. We may assume that $c \in A \setminus B$ and $d \in B \setminus A$. Therefore there exist $C, D \in \mathfrak{A}_L$ such that $c, c \vee d \in C$ and $d, c \vee d \in D$. Then $c, d, c \vee d \notin \mathbf{C}(L)$ with $c \in A \cap C$, $c \vee d \in C \cap D$ and $d \in D \cap B$. □

Let L be an OML. A subalgebra S of L is said to be a *full subalgebra* if every block of S is a block of L. Note that each $\mathbf{C}(x)$ is a full subalgebra of L for all $x \in L$ since $\mathfrak{A}_{\mathbf{C}(x)} = \{B \in \mathfrak{A}_L \mid x \in B\}$.

LEMMA 2.3. *Let* L *be an OML. If* $[0, x]$ *is path-connected* $\forall x \in L \setminus \mathbf{C}(L)$, *then* L *is path-connected.*

PROOF. Let $A, B \in \mathfrak{A}_L$. First, if $A \cap B \neq \mathbf{C}(L)$, then there exists $y \in A \cap B \setminus \mathbf{C}(L)$. Since $y, y' \notin \mathbf{C}(L)$, $[0, y]$ and $[0, y']$ are path-connected by the hypothesis. Thus $\mathbf{C}(y)$ is path-connected by proposition (1.3) since $\mathbf{C}(y) = [0, y] \oplus [0, y']$. Thus A and B are path-connected in $\mathbf{C}(y)$ and therefore in L since $\mathbf{C}(y)$ is a full subalgebra of L. Second, if $A \cap B = \mathbf{C}(L)$ and $A \cup B \leq L$, then A and B are path-connected. Finally, if $A \cap B = \mathbf{C}(L)$ and $A \cup B \not\leq L$, then there exist $C, D \in \mathfrak{A}_L$ such that

$A \cap C \neq \mathbf{C}(L)$, $C \cap D \neq \mathbf{C}(L)$ and $D \cap B \neq \mathbf{C}(L)$ by lemma (2.2). Thus A and B are path-connected by a concatenated path by the first case.□

We are ready to prove one of our main theorems.

THEOREM 2.4. *Every chain-finite OML is path-connected.*

PROOF. Let L_1 be a chain-finite OML which is nonpath-connected. So L_1 is non-Boolean. We construct an infinite sequence $x_1 > x_2 > x_3 > \cdots$ by induction. There exists at least one $x_1 \in L_1 \setminus \mathbf{C}(L_1)$ such that $L_1[0, x_1]$ is not path-connected by lemma (2.3) since L_1 is not path-connected. Let $L_2 = L_1[0, x_1]$. Assume that there exist $x_i \in L_i \backslash \mathbf{Cen}(L_i)$ such that $L_i[0, x_i]$ is not path-connected in L_i for $1 \leq i \leq n$ and $x_1 > x_2 > \cdots . > x_{n-1} > x_n$. Let $L_{n+1} = L_n[0, x_n]$. Then there exists $x_{n+1} \in L_{n+1} \backslash \mathbf{Cen}(L_{n+1})$ such that $L_{n+1}[0, x_{n+1}]$ is not path-connected in L_{n+1} and $x_1 > x_2 > \cdots > x_n > x_{n+1}$ by lemma (2.3) since L_{n+1} is not path-connected. Thus we have an infinite sequence $x_1 > x_2 > x_3 > \cdots .$ contradicting the chain-finiteness of L. This completes the proof. □

We need the following lemma to prove theorem (2.6).

LEMMA 2.5. *Every OML L containing a finite block has a unique orthogonal decomposition $L = L_0 \oplus L_1 \oplus \cdots \oplus L_n$ $(0 \leq n)$ where L_0 is a Boolean algebra and $L_1, L_2, \cdots, L_n$ are irreducible OMLs each containing a finite block.*

PROOF. Let A be a finite block of L. Then there exist only finitely many distinct minimal non-zero elements $c_1, c_2, \cdots, c_k \in \mathbf{C}(L)$ since $\mathbf{C}(L) \subseteq A$, and $L = \bigoplus_{i=1}^{k}[0, c_i]$. Let $I = \{i \mid c_i \text{ is an atom of } L\}$, $L_0 = \bigoplus_{i \in I}[0, c_i]$ and $J = \{i \mid 1 \leq i \leq k,\ i \notin I\}$. Then $L = L_0 \oplus \bigoplus_{j \in J}[0, c_j]$ where L_0 is a Boolean algebra since each $[0, c_i] \subset \mathbf{C}(L)$ $(i \in I)$ and $[0, c_j]$ is an irreducible OML containing a finite block. Let $n = |J|$. This completes the proof. □

THEOREM 2.6. *Each finite block of a non-Boolean OML L is path-connected with at least one other block of L.*

PROOF. We may assume that $|L| \geq 4$, and that L is an irreducible OML containing a finite block A by proposition (1.3) and lemma (2.5). We will prove the claim by induction on the cardinality of A. If $|A| = 2^2$, then A is an horizontal summand of L so that A is path-connected with each block in L. Assume every finite block A with $|A| \leq 2^n$ $(n \geq 2)$ of any non-Boolean OML is path-connected with at least one other block of that OML. Let $|A| = 2^{n+1}$. Let us show that A is path-connected with at least one other block of L. First, if $A \cap B = \{0, 1\}$ for each $B \in \mathfrak{A}_L \setminus \{A\}$, then we claim that $\bigcup \mathfrak{A}_L \setminus \{A\}$ is a subalgebra of L. Let $M = \bigcup \mathfrak{A}_L \setminus \{A\}$. Suppose M is not a subalgebra of L, then there exist two distinct element a, b in M such that $a \vee b \notin M$ and $a \vee b \in A$. Since $0 < a < a \vee b < 1$, there exist a block C containing a and $a \vee b$. Thus $A \neq C$ since $a \in C \setminus A$, and $C \subset M$. Therefore $a \vee b \in A \cap C \setminus \{0, 1\}$. This contradicts the assumption $A \cap B = \{0, 1\}$ for each $B \in \mathfrak{A}_L \setminus \{A\}$. Thus A is a horizontal summand of L. Therefore we may assume $A \cap B \neq \{0, 1\}$ for some $B \in \mathfrak{A}_L \setminus \{A\}$. Let $x \in A \cap B \setminus \{0, 1\}$. Then $\mathbf{C}(x) = [0, x] \oplus [0, x']$ and $A = A[0, x] \oplus A[0, x']$. If $[0, x]$ is Boolean, then $[0, x']$ is path-connected by induction hypothesis since $|[0, x']| \leq 2^n$. Thus $\mathbf{C}(x) = [0, x] \oplus [0, x']$ is path-connected by proposition (1.3). Therefore we may assume that $[0, x]$ is non-Boolean. Then $A[0, x]$ is path-connected with another block $D \in \mathfrak{A}_{\mathbf{C}(x)[0,x]}$ by the induction hypothesis since $|A[0, x]| \leq 2^n$. Therefore $A = A[0, x] \oplus A[0, x']$ is path-connected with $D \oplus A[0, x'] \in \mathfrak{A}_{\mathbf{C}}(x)$ in $\mathbf{C}(x)$ by proposition (1.3). Thus $A = A[0, x] \oplus A[0, x']$ is path-connected with $D \oplus A[0, x']$ in L since $\mathbf{C}(x)$ is a full subalgebra of L. This completes the proof. □

We have the following class of path-connected OMLs with a finite condition.

THEOREM 2.7. *Every OML L with uniformly finite sites is path-connected.*

PROOF. We may assume that $|\mathfrak{A}_L| \geq 2$. We will prove the claim by induction on the maximum cardinality 2^n of $|A \cap B|$ for all distinct blocks A, B of L. Let L be an OML such that $|A \cap B| \leq 2^n$ for all distinct blocks A, B in L. If $n = 1$, then L is path-connected since L is a horizontal sum of Boolean algebras. Assume $n > 1$, and assume that every OML such that $|A \cap B| \leq 2^{n-1}$ for all distinct blocks A, B of that OML is path-connected. If L is an OML such that $|A \cap B| \leq 2^n$ for all distinct blocks A, B of L. First, assume A, B to be two distinct blocks of L such that $A \cap B \neq \mathbf{C}(L)$. If $A \cup B \leq L$, then A and B are path-connected. Thus we may assume that $A \cup B \nleq L$. Let $x \in (A \cap B) \setminus \mathbf{C}(L)$. Then $[0, x]$ and $[0, x']$ are path-connected by inductive hypothesis since $|C \cap D| \leq 2^{n-1}$ for all distinct blocks C, D of $[0, x]$ and $|E \cap F| \leq 2^{n-1}$ for all distinct blocks E, F of $[0, x']$. Thus $\mathbf{C}(x) = [0, x] \oplus [0, x']$ is path-connected by proposition (1.4). Hence A and B are path-connected in $\mathbf{C}(x)$, and therefore in L since $\mathbf{C}(x)$ is a full subalgebra of L. Finally, assume $A \cap B = \mathbf{C}(L)$. If $A \cup B \leq L$, then A and B are path-connected. Now we may assume that $A \cup B \nleq L$ and $A \cap B = \mathbf{C}(L)$. Then there exist $G, H \in \mathfrak{A}_L$ such that $A \cap G \neq \mathbf{C}(L)$, $G \cap H \neq \mathbf{C}(L)$, and $H \cap B \neq \mathbf{C}(L)$ by lemma (2.2). Thus A and B are path-connected by a concatenated path by the first case. This completes the proof. □

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