Comm. Korean Math. Soc. **14** (1999), No. 1, pp. 47–55

IMPLICATIVE ORDERED FILTERS OF IMPLICATIVE SEMIGROUPS

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Abstract. This paper continues the investigations of implicative semigroups and of their ordered filters which were started in general case by M. W. Chan and K. P. Shum [3]. In particular, the notion of an implicative ordered filter of an implicative semigroup is introduced. We state some equivalent conditions for an ordered filter to be an implicative ordered filter and obtain the so called extension property for implicative ordered filters.

1. Introduction

The notions of implicative semigroup and ordered filter were introduced by M. W. Chan and K. P. Shum [3]. The first is a generalization of implicative semilattice (see W. C. Nemitz [6] and T. S. Blyth [2]) and has a close relation with implication in mathematical logic and set theoretic difference (see G. Birkhoff [1] and H. B. Curry [4]). For the general development of implicative semilattice theory the ordered filters play an important role which is shown by W. C. Nemitz [6]. Motivated by this, M. W. Chan and K. P. Shum [3] established some elementary properties, and constructed the quotient structure of implicative semigroups via ordered filters. Y. B. Jun, J. Meng and X. L. Xin [5] discussed ordered filters of implicative semigroups. For a deep study of implicative semigroups it is undoubtedly necessary to establish more complete theory of ordered filters.

This paper continues the investigations of implicative semigroups and of their ordered filters which were started in general case by M. W.

Received May 4, 1998. Revised June 10, 1998.

1991 Mathematics Subject Classification: 20M12, 06F05, 06A06, 06A12.

Key words and phrases: implicative semigroup, ordered filter, implicative ordered filter.

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2. Preliminaries

We recall some definitions and results.

By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*) we mean a set S with a partial ordering "$\leq$" and a binary operation "$\cdot$" such that for all $x, y, z \in S$, we have:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(2) $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$,

(3) $x \cdot y \leq x$ and $x \cdot y \leq y$.

An n.p.o. semigroup $(S; \leq, \cdot)$ is said to be *implicative* if there is an additional binary operation $* : S \times S \to S$ such that for any elements x, y, z of S,

(4) $z \leq x * y$ if and only if $z \cdot x \leq y$.

The operation $*$ is called *implication.* From now on, an implicative n.p.o. semigroup will be called simply an *implicative semigroup.*

An implicative semigroup $(S; \leq, \cdot, *)$ is said to be *commutative* if it satisfies

(5) $x \cdot y = y \cdot x$ for all $x, y \in S$,

that is, $(S, \cdot)$ is a commutative semigroup.

In any implicative semigroup $(S; \leq, \cdot, *)$, the following hold: for every $x, y \in S$,

(i) $x * x = y * y$

(ii) $x * x$ is the greatest element, written 1, of $(S, \leq)$.

PROPOSITION 2.1 ([3; Theorem 1.4]). *Let S be an implicative semigroup. Then for every $x, y, z \in S$, the following hold:*

(6) $x \leq 1$, $x * x = 1$, $x = 1 * x$,

(7) $x \leq y * (x \cdot y)$,

(8) $x \leq x * x^2$,

(9) $x \leq y * x$,

(10) *if* $x \leq y$ *then* $x * z \geq y * z$ *and* $z * x \leq z * y$,

(11) *$x \leq y$ if and only if $x * y = 1$,*
(12) $x * (y * z) = (x \cdot y) * z$,
(13) *if S is commutative then $x * y \leq (s \cdot x) * (s \cdot y)$ for all s in S.*

DEFINITION 2.2 ([3; Definition 2.1]). Let S be an implicative semigroup and let F be a non-empty subset of S. Then F is called an *ordered filter* of S if

(F1) $x \cdot y \in F$ for every $x, y \in F$, that is, F is a subsemigroup of S.
(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

The following result gives an equivalent condition of an ordered filter.

PROPOSITION 2.3 ([5; Proposition 2]). *Suppose S is an implicative semigroup. Then a non-empty subset F of S is an ordered filter if and only if it satisfies the following conditions:*
(F3) $1 \in F$,
(F4) *$x * y \in F$ and $x \in F$ imply $y \in F$.*

Now we note important elementary properties of a commutative implicative semigroup, which follows from (5), (6) and (12).

OBSERVATION. *If S is a commutative implicative semigroup, then for any $x, y, z \in S$,*
(14) $x * (y * z) = y * (x * z)$.
(15) $y * z \leq (x * y) * (x * z)$.
(16) $x \leq (x * y) * y$.

3. Implicative ordered filters

In the first place, we give an equivalent condition of an ordered filter.

PROPOSITION 3.1. *Let S be a commutative implicative semigroup and let F be a non-empty subset of S. Then F is an ordered filter if and only if it satisfies for all $x, y \in F$ and $z \in S$:*
(F5) *$x \leq y * z$ implies $z \in F$.*

PROOF. Assume F is an ordered filter and $x \leq y * z$ for all $x, y \in F$ and $z \in S$. Then $x * (y * z) = 1 \in F$. Since $x, y \in F$, it follows from (F4) that $z \in F$.

Conversely, suppose F satisfies (F5). Since $x \leq x * 1$ for all $x \in F$, we have $1 \in F$ by (F5). Let $x * y \in F$ and $x \in F$. Note that $x \leq (x * y) * y$

by (16). Using (F5) we get $y \in F$. Therefore F is an ordered filter, which completes the proof. □

DEFINITION. Let S be an implicative semigroup. A non-empty subset F of S is called an *implicative ordered filter* of S if it satisfies (F3) and

(I) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$

for all $x, y, z \in S$.

EXAMPLE 3.2. Let $S := \{1, a, b, c, d, 0\}$ be a set with Cayley tables (Tables 1 and 2) and Hasse diagram (Figure 1) as follows:

$\cdot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

Table 1

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Table 2

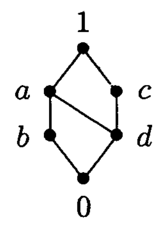

Figure 1

It is easy to see that $(S; \leq, \cdot, *)$ is an implicative semigroup, and $F := \{1, a, b\}$ is an implicative ordered filter of S.

PROPOSITION 3.3. *Every implicative ordered filter is an ordered filter.*

PROOF. Let F be an implicative ordered filter of an implicative semigroup S. Let $x * y \in F$ and $x \in F$. Then $1 * (x * y) \in F$ and $1 * x \in F$ by (6), which imply that $y = 1 * y \in F$. This completes the proof. □

REMARK 3.4. The converse of Proposition 3.3 is not true as is shown in the following example.

EXAMPLE 3.5. Let S be the implicative semigroup in Example 3.2. Then $\{1\}$ is an ordered filter of S, but not an implicative ordered filter, since $d*(a*0) = d*d = 1 \in \{1\}$ and $d*a = 1 \in \{1\}$, but $d*0 = a \notin \{1\}$.

Let S be an implicative semigroup. For any $a \in S$, we define

$$F(a) := \{x \in S | a \leq x\}.$$

Clearly, $1 \in F(a)$ and $a \in F(a)$.

REMARK 3.6. In general, $F(a)$ is not an ordered filter as shown in the following example.

EXAMPLE 3.7. Let S be the implicative semigroup in Example 3.2. Then $F(a) = \{1, a\}$ is not an ordered filter of S, since $a * b \in F(a)$ and $a \in F(a)$, but $b \notin F(a)$.

Using Proposition 2.3, we give an equivalent condition for the set $F(a)$ to be an ordered filter: Let S be an implicative semigroup. Then $F(a)$ is an ordered filter of S for all $a \in S$ if and only if $z \leq x * y$ and $z \leq x$ imply $z \leq y$ for all $x, y, z \in S$.

LEMMA 3.8. *Let S be an implicative semigroup. Then $F(a)$ is an ordered filter of S for all $a \in S$ if and only if $\{1\}$ is an implicative ordered filter of S.*

PROOF. Assume that $F(a)$ is an ordered filter of S for all $a \in S$. Let $x * (y * z) \in \{1\}$ and $x * y \in \{1\}$. Then $x \leq y * z$ and $x \leq y$. It follows that $x \leq z$ and so $x * z = 1 \in \{1\}$. Hence $\{1\}$ is an implicative ordered filter of S.

Conversely, suppose $\{1\}$ is an implicative ordered filter of S. Let $x*y \in F(a)$ and $x \in F(a)$. Then $a*(x*y) = 1 \in \{1\}$ and $a*x = 1 \in \{1\}$, which imply that $a * y \in \{1\}$. Therefore $a * y = 1$, i.e., $a \leq y$. This means $y \in F(a)$, and the proof is complete. □

Let F be an ordered filter of an implicative semigroup S and let $a \in S$. Define $F_a := \{x \in S | a * x \in F\}$. Note that $F_1 = F$.

REMARK 3.9. For an ordered filter F of an implicative semigroup S, the set F_a may not be an ordered filter of S for some $a \in S$.

EXAMPLE 3.10. Let S be the implicative semigroup in Example 3.2. Consider an ordered filter $F := \{1, c\}$. Then $F_a := \{1, a, c, d\}$ is not an ordered filter, since $a * b = a \in F_a$ and $a \in F_a$, but $b \notin F_a$.

By using the set F_a, we state an equivalent condition for an ordered filter to be an implicative ordered filter.

THEOREM 3.11. *Let F be an ordered filter of an implicative semigroup S. Then F is an implicative ordered filter if and only if for any $a \in S$, the set F_a is an ordered filter of S.*

PROOF. Assume that F is an implicative ordered filter of S. Clearly, $1 \in F_a$. Let $x * y \in F_a$ and $x \in F_a$. Then $a * (x * y) \in F$ and $a * x \in F$. Since F is an implicative ordered filter, it follows that $a * y \in F$, i.e., $y \in F_a$. Hence F_a is an ordered filter of S.

Conversely, suppose F_a is an ordered filter for all $a \in S$. Let

$$x * (y * z) \in F \text{ and } x * y \in F.$$

Then $y * z \in F_x$ and $y \in F_x$, which imply that $z \in F_x$, i.e., $x * z \in F$. Hence F is an implicative orderd filter. This completes the proof. □

LEMMA 3.12. *Let S be a commutative implicative semigroup and let F be a non-empty subset of S such that*

(F3) $1 \in F$,

(F6) $x * (y * (y * z)) \in F$ *and* $x \in F$ *imply* $y * z \in F$

for all $x, y, z \in S$. Then F is an implicative ordered filter of S.

PROOF. Let $x*(y*z)\in F$ and $x*y\in F$. Using (14) and (15) we have

$$x*(y*z)=y*(x*z)\le(x*y)*(x*(x*z)).$$

It follows from (F2) that $(x*y)*(x*(x*z))\in F$. Since $x*y\in F$, by (F6) we get $x*z\in F$. Therefore F is an implicative ordered filter. This completes the proof. □

LEMMA 3.13. *Let S be an implicative semigroup. If F is an implicative ordered filter of S, then it satisfies:*
(F7) $x*(x*y)\in F$ *implies* $x*y\in F$
for all $x,y\in S$.

PROOF. Straightforward. □

LEMMA 3.14. *Let F be an ordered filter of a commutative implicative semigroup S.*
(i) *If F satisfies* (F7), *then it also satisfies:*
(F8) $x*(y*z)\in F$ *implies* $(x*y)*(x*z)\in F$
for all $x,y,z\in S$.
(ii) *If F satisfies* (F8), *then it also satisfies* (F6).

PROOF. (i) Let $x*(y*z)\in F$. Note that

$$\begin{aligned}&x*(y*z)\\&\le x*((x*y)*(x*z)) &&\text{[by (10) and (15)]}\\&=x*(x*((x*y)*z)). &&\text{[by (14)]}\end{aligned}$$

It follows from (F2) that $x*(x*((x*y)*z))\in F$. Using (F7) and (14) we have

$$(x*y)*(x*z)=x*((x*y)*z)\in F.$$

(ii) Assume $x*(y*(y*z))\in F$ and $x\in F$. Then $y*(y*(x*z))\in F$ by (14). It follows from (6), (14) and (F8) that

$$x*(y*z)=(y*y)*(y*(x*z))\in F.$$

Since F is an ordered filter and $x\in F$, by (F4) we have $y*z\in F$. This completes the proof. □

Combining Proposition 3.3 and Lemmas 3.12, 3.13 and 3.14, we have

THEOREM 3.15. *Let S be a commutative implicative semigroup and let F be a non-empty subset of S. Then the following are equivalent:*

(i) *F is an implicative ordered filter of S.*

(ii) *F is an ordered filter, and F satisfies* (F7).

(iii) *F is an ordered filter, and F satisfies* (F8).

(iv) *F satisfies* (F3) *and* (F6).

Finally we give the extension property for implicative ordered filters.

THEOREM 3.16. (Extension property for implicative ordered filter) *Let S be a commutative implicative semigroup and let F and G be ordered filters of S such that $F \subseteq G$. If F is an implicative ordered filter, then so is G.*

PROOF. Let $x * (y * z) \in G$. Then

$$\begin{aligned} & x * (y * ((x * (y * z)) * z)) \\ &= (x * (y * z)) * (x * (y * z)) && \text{[by (14)]} \\ &= 1 \in F. && \text{[by (6)]} \end{aligned}$$

It follows from (F8) that

$$(x * y) * (x * ((x * (y * z)) * z)) \in F,$$

whence

$$\begin{aligned} & (x * (y * z)) * ((x * y) * (x * z)) \\ &= (x * y) * ((x * (y * z)) * (x * z)) && \text{[by (14)]} \\ &= (x * y) * (x * ((x * (y * z)) * z)) \in F \subseteq G. && \text{[by (14)]} \end{aligned}$$

Since G is an ordered filter and $x * (y * z) \in G$, by (F4) we have $(x * y) * (x * z) \in G$. This shows that G satisfies (F8). Thus, by Theorem 3.15, G is an implicative ordered filter of S. This completes the proof. □

ACKNOWLEDGEMENTS. The author is highly grateful to the referees for their valuable comments and suggestions for improving the paper.

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. Vol. XXV, Providence 1967.

[2] T. S. Blyth, *Pseudo-residuals in semigroups*, J. London Math. Soc. **40** (1965), 441-454.

[3] M. W. Chan and K. P. Shum, *Homomorphisms of implicative semigroups*, Semigroup Forum **46** (1993), 7-15.

[4] H. B. Curry, *Foundations of Mathematics Logic*, McGrow-Hill, New York 1963.

[5] Y. B. Jun, J. Meng and X. L. Xin, *On ordered filters of implicative semigroups*, Semigroup Forum **54** (1997), 75-82.

[6] W. C. Nemitz, *Implicative semilattices*, Trans. Amer. Math. Soc. **117** (1965), 128-142.

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