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DERIVATIONS ON SEMIPRIME MUTATION ALGEBRAS

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ABSTRACT. In [2], the author discusses derivations of $A(p,q)$ when $A(p,q)$ is prime and $p,q \in A$ satisfy the condition $A = Ap + Aq + R$ where R is a subspace of $Z(A)$. In this paper, we consider a generalization of Theorem 1 in [2] for the semiprime case of $A(p,q)$.

1. Preliminaries

Let A be an associative algebra with multiplication xy over a field F. Let p, q be two fixed elements in A, and let $A(p,q)$ denote the algebra with multiplication

$$x \star y = xpy - yqx$$

defined on the vector space A. The resulting algebra has been called the *(p,q)-mutation of A*. For a fixed element $a \in A$, denote by $A(a)$ the algebra with multiplication $x.y = xay$, but with the same vector space as A. The algebra $A(a)$ is called *the a-homotope of A*. It is clear that $A(a)$ is also associative.

By *a derivation of A* we mean a linear map $d: A \to A$ satisfying $d(xy) = d(x)y + xd(y)$, and denote by Der A the set of derivations of A. An algebra A is called *prime* if the product of any two nonzero ideals of A is nonzero. An algebra A over F of characteristic not 2 is said to be *semiprime* if $I^2 = 0$ for any ideal I of A implies $I = 0$. A prime algebra is, in particular, semiprime.

In [2], the author discusses derivations of $A(p,q)$ when $A(p,q)$ is prime and $p,q \in A$ satisfy the condition $A = Ap + Aq + R$, where R is a subspace of $Z(A)$ and investigates isomorphisms of $A(p,q)$ to $B(a,b)$ when $B(a,b)$ is prime and $p,q \in A$ satisfy the above condition.

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In this paper, we consider derivations of semiprime mutation algebra $A(p,q)$ where A satisfies conditions similar to the above.

2. Derivations on semiprime mutation algebras

PROPOSITION 2.1. *Let A be an associative algebra over F and p,q be fixed elements of A. If $A(p,q)$ is semiprime, then so is A.*

PROOF. It is immediate that any ideal of A is an ideal of $A(p,q)$ and that if $A(p,q) \star A(p,q) \neq 0$, then $A^2 \neq 0$. Assume $A(p,q)$ is semiprime, and let I be an ideal of A such that $I^2 = 0$. Since I becomes an ideal of $A(p,q)$ and by the semiprimeness of $A(p,q)$, $I \star I = 0$ implies $I = 0$. But, clearly, $I \star I \subseteq I^2 + I^2 = 0$ which implies that $I = 0$. □

PROPOSITION 2.2. *Let A be a non commutative associative algebra with 1 over F of characteristic not 2 and $p \neq 0$ be a fixed element of A. If d is a derivation of A and $d(p) = 0$, then $d(1) = 0$.*

PROOF. Note that $1 \in Z(A)$ implies $d(1) \in Z(A)$. By hypothesis, since $d(p) = 0$ and d is a derivation of A,

$$\begin{aligned} 0 &= d(p) = d(1p1) = d(1)p1 + 1d(p)1 + 1pd(1) \\ &= d(1)p + pd(1) \\ &= 2pd(1). \end{aligned}$$

Since F is of characteristic not 2, $pd(1) = 0$ and $p \neq 0$ implies $d(1) = 0$. □

The following two propositions can be easily derived from definitions.

PROPOSITION 2.3. *Let A be a prime associative algebra with 1 over F of characteristic not 2 and p be a fixed element of A satisfying $A = Ap$. Then a linear map $d : A \to A$ with $d(p) = d(1) = 0$ is a derivation of $A(p)$ if and only if d is a derivation of A.*

PROPOSITION 2.4. *Let A be a prime associative algebra with 1 over F of characteristic not 2 and p,q be fixed elements of A satisfying $A = pAq$. Then a linear map $d : A \to A$ with $d(p) = d(q) = d(1) = 0$ is a derivation of $A(p,q)$ if and only if d is a derivation of A.*

In the following proposition, we generalize theorem 1 in [2] for the case of semiprime.

PROPOSITION 2.5. *Let A be a semiprime associative algebra with 1 over F of characteristic not 2, and p be a fixed element of A satisfying $A = Ap + R$, where R is a subspace of the center $Z(A)$ of A. Then a linear map $d : A \to A$ with $d(p) = 0$ is a derivation of $A(p)$ if and only if d is a derivation of A.*

PROOF. If d is a derivation of A, then, by Proposition 1.1, $d(p) = 0$ implies $d(1) = 0$ which means that $d \in \mathrm{Der}\ A(p)$.

Conversely, assume that $d \in \mathrm{Der}\ A(p)$ with $d(p) = 0$. For $x \in A$, let $x = up + r$, $r \in R$. Then $d(x) = d(up1) + d(r) = d(u)p + d(r)$. Note that $d(r)$ satisfies $[d(r), p] = 0$. thus $d(xy) = d(upy + ry) = d(u)py + upd(y) + d(ry)$. Since $pd(ry) = 1pd(ry) = d(pry) = d(rpy)$ and $p\{d(ry) - d(r)y - rd(y)\} = d(rpy) - d(r)py - rpd(y) = 0$, it suffices to show that $d(ry) - d(r)y - rd(y) = 0$. Denote $D_{r,y} = d(ry) - d(r)y - rd(y)$. For any $z = rp + s \in A$, $s \in R$, we have

$$D_{r,y}z[r, y] = 0. \tag{2.1}$$

A linearization of (1.1) in y gives

$$D_{r,(y+z)}z\{r, y + z\} = D_{r,y}z\{r, z\} + D_{r,z}z[r, y] = 0. \tag{2.2}$$

Hence we have

$$\{D_{r,y}z[r, z]\}t\{D_{r,z}z[r, z]\} = -\{D_{r,y}z[r, z]\}t\{D_{r,z}z[r, y]\} = 0. \tag{2.3}$$

Since A is semiprime, we have

$$D_{r,y}z[r, z] = 0. \tag{2.4}$$

Furthermore,

$$s[r, y] = [s, ry] + [r, y]s + [yr, s] \tag{2.5}$$

since $s \in R$. Hence for $z = rp + s$, $s \in R$,

$$\begin{aligned}[r, y]zD_{r,y} &= [r, y](rp + s)D_{r,y} \\ &= [r, y]sD_{r,y} = 0.\end{aligned} \tag{2.6}$$

By a linearization of (1. 6) in y, we have

$$[r, y]sD_{r,t} + [r, t]sD_{r,y} = 0. \tag{2.7}$$

It follows from this that

$$([r, t]sD_{r,y})a([r, t]sD_{r,y}) = -([r, t]sD_{r,y})a([r, y]sD_{r,t}) = 0 \tag{2.8}$$

for any $a \in A$. Since A is semiprime, we have

$$[r,t]sD_{r,t} = 0. \tag{2.9}$$

Linearizing (1. 9) gives

$$[b,t]sD_{r,y} + [r,t]sD_{b,y} = 0 \tag{2.10}$$

for some $b \in A$. Hence

$$([b,t]sD_{r,y})c([b,t]sD_{r,y}) = -([b,t]sD_{r,y})c([r,t]sD_{b,y}) = 0 \tag{2.11}$$

for some $c \in A$. Therefore we have

$$[b,t]sD_{r,y} = 0. \tag{2.12}$$

In particular,

$$\begin{aligned} & [D_{r,y},t]s[D_{r,y},t] \\ = \; & (D_{r,y}t - tD_{r,y})s[D_{r,y},t] \\ = \; & D_{r,y}(ts)[D_{r,y},t] - tD_{r,y}s[D_{r,y},t] = 0. \end{aligned} \tag{2.13}$$

From the semiprimeness of A, we have

$$[D_{r,y},t] = 0, \tag{2.14}$$

implying $D_{r,y} \in Z(A)$. But, since $[b,t]sD_{r,y} = 0$,

$$([b,t]D_{r,y})s([b,t]D_{r,y}) = 0, \tag{2.15}$$

and hence

$$[b,t]D_{r,y} = 0 \tag{2.16}$$

and

$$D_{r,y}[r,y] = 0. \tag{2.17}$$

It follows from (1.17) that

$$D_{r,y}[r,A] = 0. \tag{2.18}$$

Since, in a semiprime algebra A over F, $[x,A]A$ is an ideal of A for any $x \in A$, we have

$$D_{r,y}[r,A]AD_{r,y} = 0. \tag{2.19}$$

By the semiprimeness of A, we conclude that $D_{r,y} = 0$. □

In [1], Theorem 4.4 in Chapter IV determines derivations of prime mutation algebras $A(p,q)$ with $p \neq q$ and $Ap + Aq = A$. We consider the following.

PROPOSITION 2.6. *Let A be a noncommutative associative algebra with 1 over F, and $p, q \in A$ be fixed elements such that $A(p,q)$ is prime with $p \neq q \neq 0$ and $A = Ap + Aq + R$, $R \subseteq Z(A)$. A linear map $d : A(p,q) \to A(p,q)$ with $d(p) = d(q) = 0$ is a derivation of $A(p,q)$ if and only if d is a derivation of A.*

PROOF. If d is a derivation of A, then, with the condition $d(p) = d(q) = 0$, it is easily seen that d is a derivation of $A(p,q)$.

Conversely, assume that d is a derivation of $A(p,q)$ with $d(p) = d(q) = 0$. Since $A = Ap+Aq+R$, for any $x \in A$, we can write $x = up+vq+r$, $r \in R$. Thus we have $d(x) = d(up1) + d(vq1) + d(r) = d(u)p + d(v)q + d(r)$ since $d \in \mathrm{Der}\, A(p,q)$. Hence

$$\begin{aligned} d(xy) &= d(upy + vqy + ry) \\ &= d(u)py + upd(y) + d(v)qy + vqd(y) + d(ry) \\ &= [d(u)p + d(v)q]y + [up + vp]d(y) + d(ry) \\ &= [d(x) - d(r)]y + (x - r)d(y) + d(ry) \\ &= d(x)y + xd(y) - rd(y) + d(ry) - d(r)y \\ &= d(x)y + xd(y) \end{aligned}$$

for all $x,\ y\ \in A$. □

PROPOSITION 2.7. *Let A be a noncommutative associative algebra with 1 over F, and let $p, q \in A$ be fixed elements such that $A(p,q)$ is prime with $p \neq q$ and $A = Ap+Aq+R$, $R \subseteq Z(A)$. A linear map $d : A(p,q) \to A(p,q)$ is a derivation of $A(p,q)$ if and only if d is expressed as $d = d' + R_{d(1)}$ for some $d' \in \mathrm{Der}\, A$, where $R_{d(1)}$ is the right multiplication in A by $d(1)$.*

PROOF. It is known that $f(A,A) = 0$ for a linear map $d : A(p,q) \to A(p,q)$ if and only if $d(xpy) = d(x)py + xpd(y)$ and $d(xqy) = d(x)qy + xqd(y)$ for all $x, y \in A$ if and only if $d \in \mathrm{Der}\, A(p,q)$. (cf. [1])

Let $d \in \mathrm{Der}\, A(p,q)$. Since $A = Ap + Aq + R$, any $x \in A$ is expressed as $x = up + vq + r$, $r \in R$. Since $d \in \mathrm{Der}\, A(p,q)$,

$$\begin{aligned} d(x) &= d(up1) + d(vq1) + d(r) \\ &= d(up) + upd(1) + d(v)q + vqd(1) + d(r) \\ &= d(u)p + d(v)q + (up + vq)d(1) + d(r). \end{aligned}$$

So $d(x - r) = d(u)p + d(v)q + (x - r)d(1)$. Hence $(d - R_{d(1)})(x - r) = d(u)p + d(v)q$.

Since $xy = upy + vqy + ry$, $(x - r)y = upy + vqy$. So, we have

$$\begin{aligned} & (d - R_{d(1)})[(x-r)y] \\ = \ & (d - R_{d(1)})(xy - ry) \\ = \ & [d - R_{d(1)}](xy) - [d - R_{d(1)}](ry) \\ = \ & d(xy) - xyd(1) - d(ry) + ryd(1) \\ = \ & d(upy) + d(vqy) + d(v)qy + vqd(y) - (up + vq)yd(1) \\ = \ & (d - R_{d(1)})(x - r)y + (x - r)(d - R_{d(1)})y. \end{aligned}$$ □

Therefore, putting $x - r = z$, we see that $d - R_{d(1)} \in \text{Der } A$ and if we put $d' = d - R_{d(1)}$, then $d'(p) = (d - R_{d(1)})(p) = d(p) - pd(1) = d(1)p$ since $d \in \text{Der } A(p,q)$ and $d(p) = d(1p1) = d(1)p + pd(1)$.

Conversely, if we let $d = d' + R_{d(1)}$, then

$$\begin{aligned} d(xpy) &= d'(xpy) + xpyd(1) \\ &= d'(x)py + xd'(p)y + xpd'(y) + xpyd(1) \\ &= d'(x)py + xd(1)py + xpd'(y) + xpyd(1) \\ &= [d'(x) + xd(1)]py + xp[d'(y) + yd(1)] \\ &= d(x)py + xpd(y). \end{aligned}$$

Similarly, we have $d(yqx) = d(y)qx + yqd(x)$ and hence $d \in \text{Der } A(p,q)$.

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